

Gauss with Elementary Matrices in the SoftAge

Márcio Antonio de Faria Rosa

Campinas State University – Unicamp, Brazil

Rafael Peres

Campinas State University – Unicamp, Brazil

Daniela Pereira Mendes Peres

Campinas State University – Unicamp, Brazil, d113949@dac.unicamp.br

Abstract: Software can be employed to solve math problems and we have to cooperate with them not passively but in an active way. In this article, we present situations in which the software user should be careful in order to avoid the use of the straight commands of the software that may put a matrix in its echelon form or give straight its rank. Wrong results appear in Wolfram's Mathematica software if the matrix has algebraic entries. Using the Gaussian elimination step by step method with the help of elementary matrices is a good strategy for the software user in such cases.

Keywords: Gauss, Elementary matrices, Linear algebra, Math education, Software

Introduction

Modern teaching trends have shown us the use of software as a facilitator to learn mathematics. A software is a powerful tool that saves time and optimizes problems resolutions. Whenever we teach students how to solve linear systems by the Gauss method with a software, it is important and necessary for the students to have already learnt the concept of elementary matrices.

Linear systems and their several methods of resolution - including the method with matrices - have already been studied in basic mathematics as they are part of the high school curriculum. Therefore, we have to improve the tools offered to students to develop their study of this common type of problem and we can do it using softwares.

As shown below, elementary matrices have a fundamental role in the operations made by the software to solve a linear system by the Gaussian elimination method. Thus, we have a situation that facilitates the resolution of a linear system and that encourages students to become increasingly interested in software applications within mathematics.

Softwares can be employed and we have to cooperate with them but in an active and not passive way. Abell and Braselton (2008) used *Wolfram's Mathematica* in some linear algebra problems where the commands *RowReduce* and *MatrixRank* work well. But in the case of a Gaussian elimination, when the matrix has algebraic entries, these straight commands can lead to wrong results. Then the elementary matrices help to follow the step by step elimination.

Let's discuss how the solutions of the below linear system depend on the real k .

$$\begin{aligned}x + 3y - z + 2w &= 3, \\0x + 11y - 5z + 3w &= 1, \\2x - 5y + 2z + 1w &= 0, \\4x + y + z + 2w &= k^2, \\3x + y + 2z + 5w &= 0.\end{aligned}$$

In *Mathematica* we enter the augmented matrix as a list of lists which are its rows.

In[1]:= a = {{1, 3, -1, 2, 3}, {0, 11, -5, 3, 1}, {2, -5, 2, 1, 0}, {4, 1, 1, 2, k²}, {3, 1, 2, 5, 0}};

And reduce this matrix to its echelon form.

In[2]:= MatrixForm[a] In[3]:= MatrixForm[RowReduce[a]]

$$\text{Out[2]} := \begin{pmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 11 & -5 & 3 & 1 \\ 2 & -5 & 2 & 1 & 0 \\ 4 & 1 & 1 & 2 & k^2 \\ 3 & 1 & 2 & 5 & 0 \end{pmatrix} \quad \text{Out[3]} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[4]:= MatrixRank[a]

Out[4]:= 5

If the result was correct, the augmented matrix rank would be bigger than the *coefficient* matrix rank and this fact would not depend on *k*. Therefore, for any value of *k*, the system would not have any solution. The last equation for the reduced form would be $0x + 0y + 0z + 0w + 0t = 1$. This is not true, as we shall see. Well, instead of going through the wide gate of the straight commands, we can make the computer work with the step by step reduction. In order to do this we employ the elementary matrices and we will see that we will not be going on such a narrow gate for the elementary matrices are a good tool for the computer user.

In[5]:= vshear[i_ , λ_ , j_ , n_] := Normal[SparseArray[{{j, i} → λ, {n, n}}] + IdentityMatrix[n];

In[6]:= stretchline[i_ , λ_ , n_] := Normal[SparseArray[{{i, i} → λ - 1, {n, n}}] + IdentityMatrix[n];

In[7]:= changeline[i_ , j_ , n_] := Normal[SparseArray[{{i, j} → 1, {j, i} → 1, {i, i} → -1, {j, j} → -1}, {n, n}]] + IdentityMatrix[n];

The software already has the ‘identity matrix’ and a ‘sparse matrix’ for which we define only the nonzero entries. By adding suitable sparse matrices to the identity matrix, we can easily construct the three kinds of elementary matrices $n \times n$ as we have done above.

Now we can easily follow the step by step elimination.

In[8]:= a2 = vshear[1, -3, 5, 5] . vshear[1, -4, 4, 5] . vshear[1, -2, 3, 5] . a;

In[9]:= a3 = stretchline[2, -1/8, 5] . changeline[2, 5, 5] . a2;

In[10]:= a4 = vshear[2, -3, 1, 5] . vshear[2, 11, 3, 5] . vshear[2, 11, 4, 5] . vshear[2, -11, 5, 5] . a3;

In[11]:= {MatrixForm[a], MatrixForm[a2], MatrixForm[a3], MatrixForm[a4]}

$$\begin{pmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 11 & -5 & 3 & 1 \\ 2 & -5 & 2 & 1 & 0 \\ 4 & 1 & 1 & 2 & k^2 \\ 3 & 1 & 2 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 11 & -5 & 3 & 1 \\ 0 & -11 & 4 & -3 & -6 \\ 0 & -11 & 5 & -6 & k^2 - 12 \\ 0 & -8 & 5 & -1 & -9 \end{pmatrix}, \begin{pmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 1 & -\frac{5}{8} & \frac{1}{8} & \frac{9}{8} \\ 0 & -11 & 4 & -3 & -6 \\ 0 & -11 & 5 & -6 & k^2 - 12 \\ 0 & 11 & -5 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \frac{7}{8} & \frac{13}{8} & -\frac{3}{8} \\ 0 & 1 & -\frac{5}{8} & \frac{1}{8} & \frac{9}{8} \\ 0 & 0 & -\frac{23}{8} & -\frac{13}{8} & \frac{51}{8} \\ 0 & 0 & -\frac{15}{8} & -\frac{37}{8} & k^2 + \frac{91}{8} \\ 0 & 0 & \frac{15}{8} & \frac{13}{8} & -\frac{91}{8} \end{pmatrix}$$

In[12]:= a5 = stretchline[3, 8/15, 5] . changeline[3, 5, 5] . a4;

In[13]:= a6 = vshear[3, 23/8, 5, 5] . vshear[3, 15/8, 4, 5] . vshear[3, 5/8, 2, 5] . vshear[3, -7/8, 1, 5] . a5;

In[14]:= a7 = stretchline[4, 15/13, 5] . changeline[4, 5, 5] . a6;

In[15]:= a8 = vshear[4, 3, 5, 5] . vshear[4, -13/15, 3, 5] . vshear[4, -2/3, 2, 5] . vshear[4, -13/15, 1, 5] . a7;

In[16]:= {MatrixForm[a5], MatrixForm[a6], MatrixForm[a7], MatrixForm[a8]}

$$\begin{pmatrix} 1 & 0 & \frac{7}{8} & \frac{13}{8} & -\frac{3}{8} \\ 0 & 1 & -\frac{5}{8} & \frac{1}{8} & \frac{9}{8} \\ 0 & 0 & 1 & \frac{13}{15} & -\frac{91}{15} \\ 0 & 0 & -\frac{15}{8} & -\frac{37}{8} & k^2 + \frac{3}{8} \\ 0 & 0 & -\frac{23}{8} & -\frac{13}{8} & \frac{51}{8} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \frac{13}{15} & \frac{74}{15} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{8}{3} \\ 0 & 0 & 1 & \frac{13}{15} & -\frac{91}{15} \\ 0 & 0 & 0 & -3 & k^2 - 11 \\ 0 & 0 & 0 & \frac{13}{15} & -\frac{166}{15} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \frac{13}{15} & \frac{74}{15} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{8}{3} \\ 0 & 0 & 1 & \frac{13}{15} & -\frac{91}{15} \\ 0 & 0 & 0 & 1 & -\frac{166}{13} \\ 0 & 0 & 0 & -3 & k^2 - 11 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 16 \\ 0 & 1 & 0 & 0 & \frac{76}{13} \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -\frac{166}{13} \\ 0 & 0 & 0 & 0 & k^2 - \frac{641}{13} \end{pmatrix}$$

For $k = \pm\sqrt{641/13}$ the system is possible and has only one solution: $x = 16, y = 76/13, z = 5, w = -166/13$. *RowReduce* and *RowRank* commands failed and this can be well understood if we see how *Mathematica* simplifies the solution.

$$\text{In[17]:= Simplify} \left[\frac{-\frac{641}{13} + k^2}{-\frac{641}{13} + k^2} \right]$$

$$\text{Out[17]:= 1}$$

This strategy is not so good for some specific problems as we have seen.

Other problems also need the step by step procedure such as Lipschutz and Lipson (2004) 3.14 (p. 97), 3.52 (p. 111) and 3.56 (p. 112). To these and to a wider class of problems, the elementary matrices enhance the computer applications for a Gaussian elimination.

Let's work problem 4.117 in Lipschutz and Lipson (2004, p.166), "Consider the following subspaces of \mathbb{R}^5 :

$$U = \text{Span}\{\{1, -1, -1, -2, 0\}, \{1, -2, -2, 0, -3\}, \{1, -1, -2, -2, 1\}\}$$

$$V = \text{Span}\{\{1, -2, -3, 0, -2\}, \{1, -1, -3, 2, -4\}, \{1, -1, -2, 2, -5\}\}$$

- (a) Find two homogeneous systems whose solution spaces are U and V , respectively.
- (b) Find a basis and the dimension of $U \cap V$."

$$\text{In[18]:= } u = \text{RowReduce}\{\{1, -1, -1, -2, 0\},$$

$$\{1, -2, -2, 0, -3\}, \{1, -1, -2, -2, 1\}\};$$

$$\text{In[19]:= } u2 = \{u[[1]], u[[2]], u[[3]], \{x, y, z, w, t\}\};$$

$$\text{In[20]:= MatrixForm}[u2]$$

$$\text{Out[20]:= } \begin{pmatrix} 1 & 0 & 0 & -4 & 3 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ x & y & z & w & t \end{pmatrix}$$

$$\text{In[21]:= } v = \text{RowReduce}\{\{1, -2, -3, 0, -2\},$$

$$\{1, -1, -3, 2, -4\}, \{1, -1, -2, 2, -5\}\};$$

$$\text{In[22]:= } v2 = \{v[[1]], v[[2]], v[[3]], \{x, y, z, w, t\}\};$$

$$\text{In[23]:= MatrixForm}[v2]$$

$$\text{Out[23]:= } \begin{pmatrix} 1 & 0 & 0 & 4 & -9 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ x & y & z & w & t \end{pmatrix}$$

It follows that U and V are three dimensional subspaces of \mathbb{R}^5 . For each of the reduced generator matrices we have added a row vector $\{x, y, z, w, t\}$ being the first of them an element of U , if the ranks of u and $u2$ are equal. Similarly the ranks of v and $v2$ are equal, if the second is an element of V . Since $u2$ and $v2$ have algebraic entries, we cannot apply the *RowReduce* command to them, we have to employ the elementary matrices.

In[24]:= u3 = vshear[3,-z,4,4].vshear[2,-y,4,4].vshear[1,-x,4,4].u2;

In[25]:= MatrixForm[u3]

$$\text{Out[25]} := \begin{pmatrix} 1 & 0 & 0 & -4 & 3 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & w + 4x + 2y & t - 3x - 4y + z \end{pmatrix}$$

In[26]:= MatrixForm[Grad[{u3[[4,4]], u3[[4,5]]}, {x,y,z,w,t}]]

$$\text{Out[26]} := \begin{pmatrix} 4 & 2 & 0 & 1 & 0 \\ -3 & -4 & 1 & 0 & 1 \end{pmatrix}$$

In[27]:= v3 = vshear[3,-z,4,4].vshear[2,-y,4,4].vshear[1,-x,4,4].v2;

In[28]:= MatrixForm[v3]

$$\text{Out[28]} := \begin{pmatrix} 1 & 0 & 0 & 4 & -9 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & w - 4x - 2y & t + 9x + 2y + z \end{pmatrix}$$

In[29]:= MatrixForm[Grad[{v3[[4,4]], v3[[4,5]]}, {x,y,z,w,t}]]

$$\text{Out[29]} := \begin{pmatrix} -4 & -2 & 0 & 1 & 0 \\ 9 & 2 & 1 & 0 & 1 \end{pmatrix}$$

The gradient operator helps us get coefficients from linear combinations. The condition for equal ranks of u and $u2$ is that the fourth row of $u3$ is null, analogously, for equal ranks of v and $v2$, the fourth row of $v3$ has to be null. From these conditions, we can obtain homogenous linear systems for U and V , respectively

$$\begin{aligned} 4x + 2y + 0z + 1w + 0t = 0, & \quad 4x - 2y + 0z + 1w + 0t = 0, \\ -3x - 4y + 1z + 0w + 1t = 0, & \quad 9x + 2y + 1z + 0w + 1t = 0, \end{aligned}$$

and for the intersection $U \cap V$, the homogeneous system has four equations.

In[30]:= uv = Grad[{u3[[4,4]], u3[[4,5]], v3[[4,4]], v3[[4,5]]}];

In[31]:= uv2 = RowReduce[uv];

In[32]:= {MatrixForm[uv], MatrixForm[uv2]}

$$\text{Out[32]} := \left\{ \left(\begin{pmatrix} 4 & 2 & 0 & 1 & 0 \\ -3 & -4 & 1 & 0 & 1 \\ -4 & -2 & 0 & 1 & 0 \\ 9 & 2 & 1 & 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 1 & -\frac{2}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \right\}$$

Rank 3 means that $U \cap V$ is a two dimensional subspace of \mathbb{R}^5 . From the reduced form of the homogeneous system we can employ z and t as free variables.

In[33]:= Solve[uv2.{x,y,z,w,t} == {0,0,0,0}, {x,y,w}]

Out[33]:= {{x -> $\frac{1}{5}(-t - z)$, y -> $\frac{2(t+z)}{5}$, w -> 0}}

In[34]:= {%%[[1,1,2]], %%[[1,2,2]], z, %%[[1,2,2]], t};

In[35]:= Transpose[Grad[%, {t, z}]]//MatrixForm

Therefore, we have the following basis for $U \cap V$,

$$\{-1/5, 2/5, 0, 0, 1\}, \{-1/5, 2/5, 1, 0, 0\}.$$

Recommendations

We recommend this type of work to engineering students, who need to learn how to work with linear systems using the software.

In our teaching experience when students do calculus activities using the software and geometric analysis they tend to have huge learning gains.

Acknowledgements

We are grateful for the fine and kind review done by Teresa Christina Waeny Pessoa de Mello Dias.

References

- Abel, L. M., & Braselton, J. P. (2008). *Mathematica by Example* (4th ed.). San Diego, California: Elsevier Academic Press.
- Lipschutz S., Marc L. L., (2004). *Schaum's Outline of Theory and Problems of Linear Algebra* (3th ed.). Pennsylvania, NY: The McGraw-Hill Companies.