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Continuity: From Intuition to Formalization – A Comprehensive Study

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Abstract

This study examines the difficulties students face in understanding the concept of continuity and its connection to key theorems such as the Fundamental Theorem of Algebra. The initial hypothesis suggests that the challenge lies not in the intuitive notion of continuity, but in its mathematical formalization, which is defined through limits within a topological framework. Although students approach problems continuously and intuitively, many struggle to translate their geometric reasoning into formal algebraic representations. To support this transition, the “Robot Trajectories” activity was designed using a Problem-Based Learning (PBL) approach. In this task, 79 students, organized into 29 teams, analyzed different paths a robot could take to reach a target point. The trajectories, represented as piecewise polynomial functions, required students to identify the optimal route and link it to the corresponding function rule, often considering obstacles along the way. While most teams relied on geometric intuition to approach the problem, only a few succeeded in achieving a solid algebraic formalization. These results highlight the gap between intuitive understanding and mathematical formalization, emphasizing the need for support for this transition in engineering calculus education.

Introduction

Understanding why undergraduate students struggle with the concept of continuity in Differential and Integral Calculus was a central objective of this study. In previous studies, the authors found evidence that part of this difficulty lies in the transition from an intuitive perception of continuity to its algebraic formalization (Bolaños Evia, G. R., Domínguez Albino, C., Hernández Grovas, F. T., Romero Rosaldo, Y., Salvador Montes, J., Vega Lara, P., 2025). This led us to design a Problem-Based Learning Realistic scenario to determine whether students could apply their intuitive notion of continuity to solve a realistic problem and subsequently formalize that notion algebraically through geometric and algebraic manipulation of concepts like polynomial roots, linear functions, and piecewise functions. This analysis was conducted using the horizontal mathematization theory of Kirie and Pieren (Pirie, S. E. ,Kieren, T. E., 1994). and the vertical mathematization theory in the context of Realistic Mathematics Education (RME), developed by the Freudenthal Institute in the Netherlands. (Freudenthal, H. 1991), (Bressan, A., Gallego, M., Pérez,S., Zolkower, B. 2016), (van den Heuvel-Panhuizen,M. 2020).

For this purpose, we propose a didactic approach based on visual representation, which is more closely aligned with intuition (Reference), and gradually guides students toward formalization through a series of steps. The "Robot Trajectories" activity presents a contextualized problem requiring students to model trajectories on a Cartesian plane using polynomial and linear functions. According to the Pirie and Kieren model, this strategy fosters deeper and more meaningful learning. This approach allowed the evaluation of students' conceptual understanding of continuity and the observation of how they develop their mathematical strategies and arguments in problem-solving situations. For engineering automation processes, formalization is crucial, as it provides the means to articulate precise instructions necessary for programming for implementation in artificial intelligence systems.

Methodology

The study was conducted with 79 undergraduate engineering students organized into teams based on the principles of Problem-Based Learning. The use of technology was restricted. The primary reason for this restriction was to encourage students to identify and construct the required trajectories based on polynomial roots, piecewise functions, and continuity. Use of technology would have enabled them to obtain these trajectories automatically by simply identifying a few key points.

The level of students' horizontal and vertical mathematization was measured using the Pirie and Kieren model. Through statistical tests such as Fisher's exact test, relevant associations were identified between the types of justifications used and the accuracy of the responses. This approach made it possible to assess the students' conceptual understanding of continuity and observe how they developed their mathematical strategies and reasoning in problem-solving situations.

The distribution of teams according to academic program and the gender composition of their members—denoted as W for teams with most women, M for teams with most men, and E for teams with an equal number of men and women—is as follows:

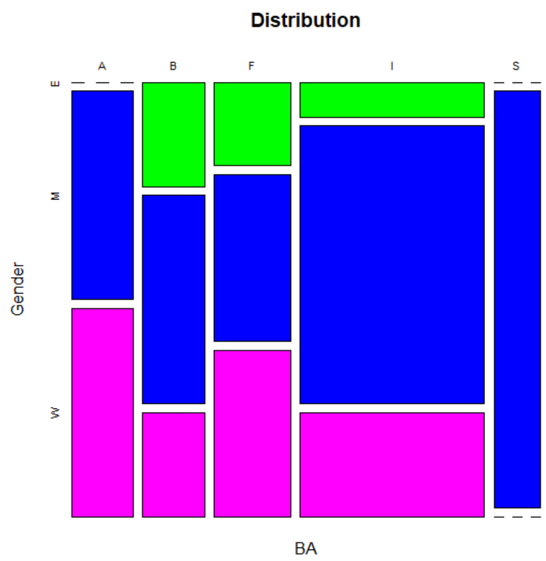

Figure 1. Distribution of Teams

The instrument is a realistic problem-based learning scenario in which a robot must move from a starting point to a target using one of several given polynomial trajectories, followed by a linear trajectory, while avoiding obstacles. It consists of four items.

Robot Trajectories

A robot needs to reach its target. To achieve this, it is necessary to represent the possible trajectories it can follow on a Cartesian plane. The robot is indicated by the green point with coordinates (1,2), Its goal is the magenta star at the coordinates. (2.5,-2). Due to obstacles represented by red diamonds and additional constraints, we have found a method to help the robot reach its goal. First, the robot must follow a polynomial trajectory to the X-axis, drawn in a yellow line, and then move in a straight line. The five possible polynomial trajectories are shown in the following figure.

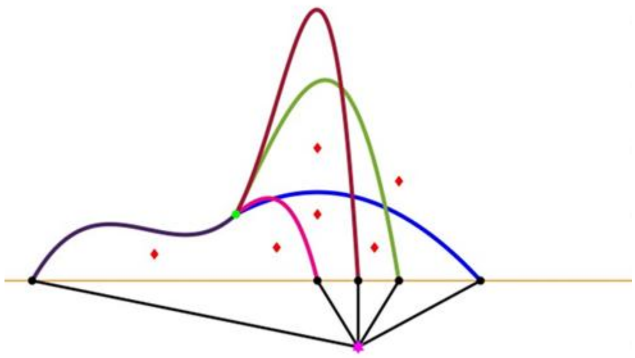

Figure 2. Possible Trajectories

Which trajectory do you think is best for the robot? Indicate it by retracing it with a pen on the previous figure. Justify your answer.

This question focuses on an intuitive response that considers the criteria used for selection. Responses were grouped into the following categories: height on the y-axis, curvature, arc length, directness, obstacles, polynomial degree, simplicity, and speed. Based on these responses, the level of horizontal mathematization was measured.

1. The trajectories correspond to the graphs of the following functions:
    a. $f(x) = \frac{-2}{3}x^2 + \frac{8}{3}x$
    b. $g(x) = -2x^3 + 7x^2 - 3x$
    c. $h(x) = -x^4 + 2x^3 - x^2 + 2x$
    d. $t(x) = -x^5 + \frac{5}{2}x^4 - \frac{1}{3}x + \frac{5}{6}$
    e. $w(x) = \frac{4}{5}x^3 + \frac{1}{5}x^2 - \frac{1}{2}x + \frac{3}{2}$

    Circle the function corresponding to the graph you selected in the previous question.

In this question, we aimed to explore the association between geometric intuition and algebraic knowledge.

Although no justification was required, some teams evaluated the trajectory at specific points to identify the corresponding polynomial. Other teams assumed the selected trajectory was a parabola but did not verify it.

2. The functions that correspond to each of the trajectories are as follows:
    a. $f(x) = \frac{-2}{3}x^2 + \frac{8}{3}x$
    b. $g(x) = -2x^3 + 7x^2 - 3x$
    c. $h(x) = -x^4 + 2x^3 - x^2 + 2x$
    d. $t(x) = -x^5 + \frac{5}{2}x^4 - \frac{1}{3}x + \frac{5}{6}$
    e. $w(x) = \frac{4}{5}x^3 + \frac{1}{5}x^2 - \frac{1}{2}x + \frac{3}{2}$

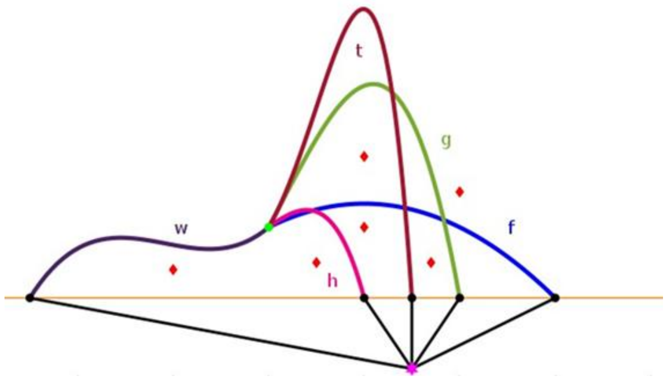

Figure 4. Possible Trajectories with Labels

Would you change your answer to question 1? Justify

This question explores whether the degree of the polynomial was a reason for students to change their initially selected trajectory and, regardless of whether they changed their choice, what arguments they used to support their decision. These arguments were classified as either geometric or algebraic and categorized as strong, moderate, or weak, depending on whether they involved the roots of polynomials and whether mathematical development was presented. Based on the justifications provided, the level of horizontal mathematization was measured.

3. If the black points are on the X-axis, can you help the robot by finding the equation of the line that joins the black point on the graph you chose in the previous question with its target? Explain how to find the line's equation to help the robot.
4. Can you define a function for the complete trajectory?

In this question, we identified whether a solution was provided and if the solution was correct or incorrect. Based on the procedures, the level of horizontal mathematization was evaluated.

By analyzing the written justifications provided by the teams, the level of mathematization they achieved was measured. Using all procedures applied throughout the scenario, the highest level of horizontal mathematization achieved was determined, and the level of vertical mathematization was also assessed.

The analysis of the mathematization level achieved by the students was conducted within the framework of the Pirie and Kieren model of Growth in mathematical understanding (Pirie, S.E., Kieren, T. E., 1994). The horizontal level consists of eight levels.

1. Primitive Knowing: The student has informal prior knowledge of the topic, acquired through everyday experience.
2. Image Making: Mental images of mathematical concepts are formed through experimentation and exploration.
3. Image Having: Confidence is gained by manipulating images without constantly reconstructing them.
4. Property Noticing: Patterns and mathematical properties within the explored concepts are identified.
5. Formalizing: Properties are organized and structured into more general mathematical principles.
6. Observing: The acquired knowledge is analyzed, and a more abstract perspective of the mathematical concept is established.
7. Structuring: The concept is recognized in different contexts, and knowledge is applied to broader situations.
8. Inventing: Advanced levels of understanding are reached, where new mathematical ideas can be created and knowledge extended.

For the analysis of horizontal mathematization, Folding Back was identified, which is a distinctive element of the mathematization theory in the context established by Pirie and Kieren. This element refers to a learner returning to earlier levels of understanding when they encounter difficulty at a higher level (Pirie, S. E., Kieren, T. E., 1994). We also considered the four levels of vertical mathematization in the context of Realistic Mathematics Education (RME), which emphasizes the progression of learners' mathematical understanding through a process of mathematization, whereby students transition from informal, contextually grounded reasoning toward increasingly formalized and abstract mathematical concepts and structures.

1. Situational Level: Students begin understanding problems in real-life or meaningful contexts. Reasoning is based on informal knowledge and interpretations grounded in the specific situation.
2. Referential Level (Model of): Context-specific strategies and models emerge. Students create representations (diagrams, tables, etc.) that refer to the original situation. Models and strategy are still closely tied to the context.
3. General Level (Model for): The previously context-specific models are now used more generally, becoming tools for reasoning in other situations. Shift from "model of a situation" to "model for a situation " broadens mathematical reasoning.
4. Formal Level: Students operate inside the domain of formal mathematics, using symbols, procedures, and abstract reasoning. Independent of the original context, with a focus on mathematical structure.

The statistical analysis was conducted using Fisher's exact test.

Results

Per question

Question 1

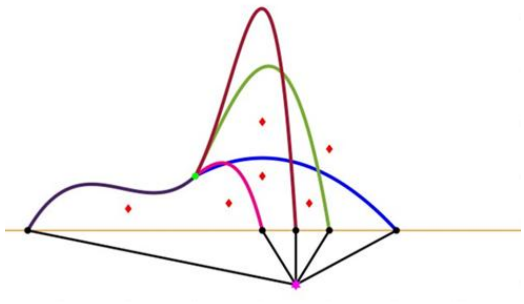

Figure 5.

Table 1.

Math. Level / Argument	Height on the y-axis	Curvature	Arc length	Direct	Obstacles	Polynomial degree	Simplicity	Speed
1	0	0	2	0	0	0	0	0
2	0	1	6	1	0	1	2	2
3	1	0	6	0	1	0	0	0
4	0	0	4	0	0	0	2	0

Responses were categorized based on the justifications provided by the teams. The majority selected the pink path, justifying their choice by stating it was the shortest. Only one or two teams chose other paths, and no team selected the black path. The maximum level of mathematization attainable for this question is level 4, which appears reasonable given that the question emphasizes geometric intuition rather than encouraging algebraic formalization. At level 2, a greater diversity of reasoning approaches was observed.

Question 2

The fact that the correct answers to this question were disclosed in Question 3 likely influenced the limited number of teams that attempted a formal procedure to determine the polynomial associated with their chosen path. Nonetheless, some teams justified their selections using polynomial roots, establishing a coordinate scale along the x-axis and assigning coordinates to the points accordingly.

Question 3

After being presented with the polynomial function associated with each trajectory, teams are required to decide whether to revise their initial selection and provide a justification. These justifications are categorized based on their geometric or algebraic nature, with particular attention given to the use of polynomial roots and the quality of their reasoning, classified as good, fair, or poor. The level of horizontal mathematization is assessed. Additionally, we examine whether there is an association between the gender composition of the teams, the type and quality of the argument, the use of roots, and the academic program.

Table 2.

	Argument	**Geometric**		**Algebraic**	
	Change	No	Yes	No	Yes
Quality	Good	11	1	5	2
	Average	1	1	0	1
	Poor	2	2	3	0

Fisher's Exact Test revealed a non-significant association between changing their selected trajectory vs the argument ($p = 0.75$), type of argument vs response quality ($p = 0.9471$). Most teams do not alter their selection, as much of the reasoning is well-founded and grounded in geometric principles.

Table 3.

	BA program	**Aero**			**Bio**			**Pharma**			**Industrial**			**Automotive**		
	Gender Group	E	M	W	E	M	W	E	M	W	E	M	W	E	M	W
Quality	Good	0	1	1	1	2	1	0	2	0	0	6	3	0	1	0
	Average	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0
	Poor	0	1	0	0	0	0	1	0	2	1	0	0	0	1	0

Fisher's Exact Test revealed a non-significant association between quality and gender group. ($p = 0.8209$), quality vs BA program ($p = 0.8021$),gender group vs BA program ($p = 0.8021$).

Table 4

	BA program	**Aero**			**Bio**			**Pharma**			**Industrial**			**Automotive**		
	Gender Group	E	M	W	E	M	W	E	M	W	E	M	W	E	M	W
Math. Level	2	0	1	0	0	1	1	0	0	0	1	2	1	0	0	0
	3	0	1	0	0	0	0	0	1	1	0	3	2	0	3	0
	4	0	0	0	0	1	0	1	0	1	0	3	0	0	0	0
	5	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0
	6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0

Fisher's Exact Test revealed a non-significant association between the BA program and math level ($p = 0.1843$)gender group and math level ($p = 0.4606$). The highest level of mathematization observed was 6, attained by only one team.

Table 5

		Response Quality	Average	Good	Poor
		Polynomial Roots			
Math. Level	2	No	0	6	1
		Yes	0	1	0
	3	No	2	6	3
		Yes	0	0	0
	4	No	0	2	3
		Yes	0	1	0
	5	No	1	2	0
		Yes	0	0	0
	6	No	0	0	0
		Yes	0	1	0

Fisher's Exact Test revealed a significant association ($p = 0.0317$) between the math level and the use of polynomial roots; the maximum level (6) was reached using polynomial roots. Most teams do not use polynomial roots. The quality of response and the use of polynomial roots are independent.

The limited use of polynomial roots may be attributed to several factors. In algebra instruction, there is often a lack of effective geometric integration. In the Robot scenario, the coordinates of the points on the x-axis are not provided; using roots requires determining these coordinates. Without technological tools, students must rely on manual calculation, which can be challenging in certain cases. Moreover, the correct placement of grouping symbols remains a common issue, even at the university level.

Question 4

In this question, students must find the equation of the line that joins the black point on the graph they chose in the previous question with its target and explain how they construct the line's equation. Solutions were classified according to correctness and type of solution. Although the equation of a line might be considered a low-difficulty task by instructors, we found further evidence that students face challenges formulating the equation correctly

Fisher's Exact Test yielded a statistically significant result (p=0.001), indicating a robust association between the solution type and response accuracy. Most teams employ polynomial roots, developing a theoretical approach that fails to formalize as an explicit function. The likely causes for this shortfall are geometric-algebraic, as discussed after the commentaries on Question 3. The highest percentage of correct responses are explicit solutions using polynomial roots, while most incorrect responses are theoretical involving roots. The mathematization level is correlated with the type of response($p = 0.002$), with higher levels achieved in explicit answers that use roots. The maximum level reached is 6, just as in question 3, but the number of teams at that level increased.

Table 6

		Type	Explicit Linear Equation Derived via Polynomial Roots	Theoretical Derivation of Linear Equation via Polynomial Root	Explicit Linear Equation Derived Independently of Polynomial Roots	Theoretical Derivation of Linear Equation Independent of Polynomial Roots
		Solution				
	2	Correct	0	0	0	0
		Incorrect	0	1	1	0
Math.		No solution	0	0	0	0
Level	3	Correct	0	0	0	0
		Incorrect	0	0	0	0
		No solution	0	0	0	3
	4	Correct	2	2	0	0
		Incorrect	3	9	1	0
		No solution	0	0	0	0
	5	Correct	2	1	0	0
		Incorrect	0	0	0	0
		No solution	0	0	0	0
	6	Correct	1	1	0	0
		Incorrect	1	0	0	0
		No solution	0	0	0	0

Question 5

Students must define a piecewise function for the complete trajectory of the robot; 24% of teams could not construct the function. The maximum level reached is 6, but the number of teams at that level increased. All the defined functions were continuous. In some cases, the segment of the line that ensured the robot would reach its target was not specified. Certain teams introduced their notation for piecewise-defined functions but failed to explain it. Additionally, some teams employed vector-valued functions, as observed in the image; not all trajectories can be described as functions of a single variable.

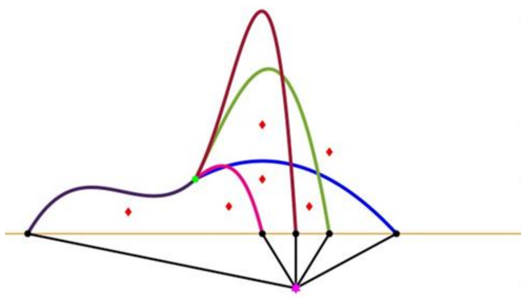

Figure 6.

Table 7

Math. Level Q5	Correct	Incorrect	No Solution
1	0	0	1
2	1	2	1
3	0	2	3
4	0	6	1
5	2	0	1
6	4	2	0

Global

An analysis of the teams' progress throughout the instrument reveals that those employing algebraic reasoning in question three provide explicit solutions, with none offering a purely theoretical response. This suggests that certain teams encountered difficulties establishing connections between geometric and algebraic representations. However, when the processes are integrated in question five, no significant differences are observed between geometric and algebraic arguments in terms of response accuracy. This indicates the presence of the folding back process between questions 3, 4, and 5, facilitating reflection on prior processes and then promoting advancement in the level of mathematization (see figure 7).

Table 8.

Type of solution at question 3		Solution question 5	Correct	Incorrect	No Solution
		Type of solution at question 4			
	Geometric	Explicit Linear Equation Derived via Polynomial Roots	2	2	3
	Geometric	Explicit Linear Equation Derived Independently of Polynomial Roots	1	4	1
	Geometric	Theoretical Derivation of Linear Equation via Polynomial Root	0	2	0
	Geometric	Theoretical Derivation of Linear Equation Independent of Polynomial Roots	0	0	3
	Algebraic	Explicit Linear Equation Derived via Polynomial Roots	0	1	1
	Algebraic	Explicit Linear Equation Derived Independently of Polynomial Roots	4	2	2
	Algebraic	Theoretical Derivation of Linear Equation via Polynomial Root	0	0	0
	Algebraic	Theoretical Derivation of Linear Equation Independent of Polynomial Roots	0	0	0

Fisher's Exact Test also reveals a statistically significant association ($p = 0.02$) between response accuracy on Questions 4 and 5.

Table 9

Function Q5 Solution Q4	Correct	Incorrect	No solution
Correct	5	1	3
Incorrect	2	10	4
No solution	0	1	3

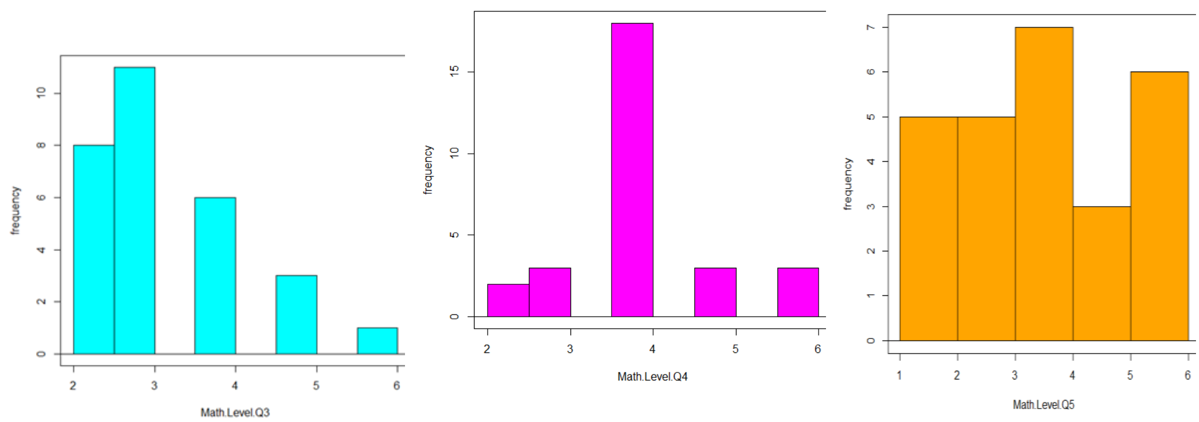

Figure 7. Progression in the level of mathematization

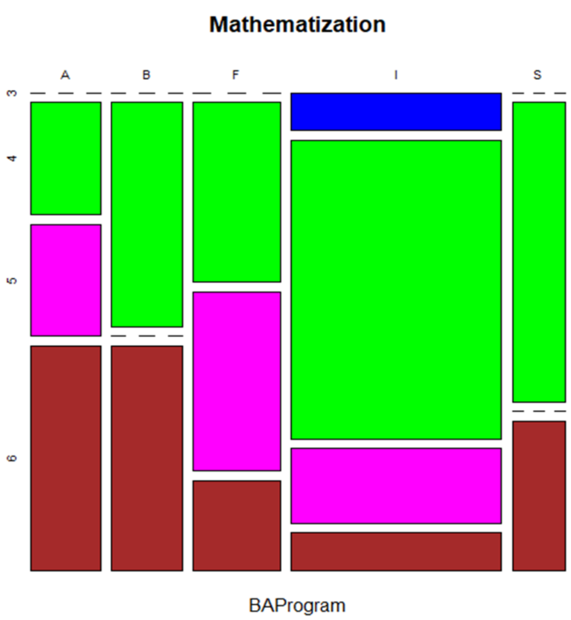

Figure 8

We calculated the highest level of mathematization achieved by each team and compared the results across the different academic programs. Fisher's Exact Test yielded a non-statistically significant result (p=0.7182), indicating independence between the BA program and the mathematization level.

Comparing the highest level of mathematization between the explicit and theoretical solutions, a difference was observed again. Fisher's Exact Test yielded a statistically significant result (p=0.036), indicating an association between the solution type and the mathematization level.

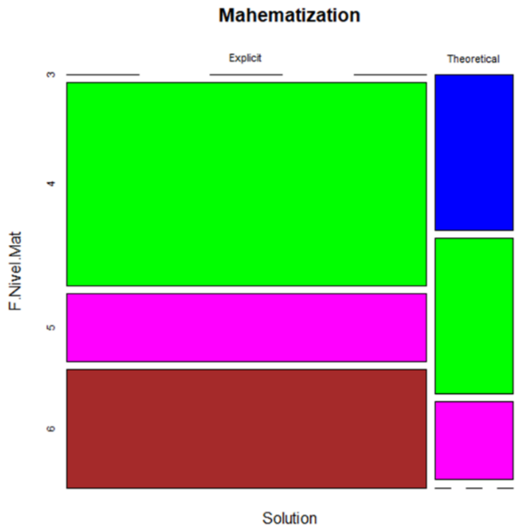

Figure 9

No significant differences were observed in the level of mathematization concerning the gender group.

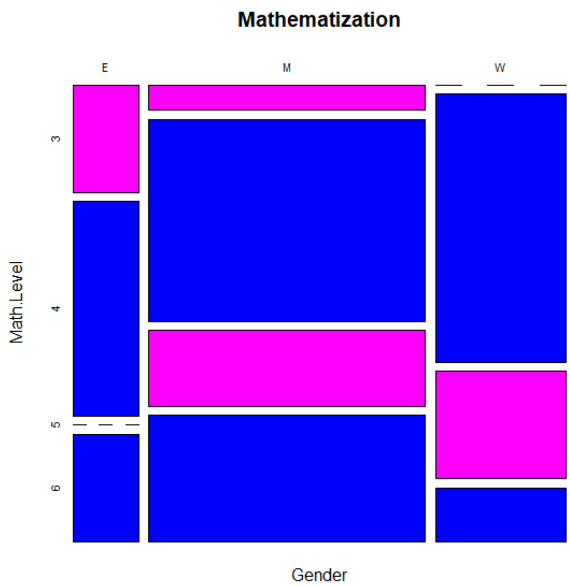

Figure 10

Although there is no significant difference between the theoretical and explicit solutions associated with gender composition, it can be observed that in groups with most women, only explicit solutions were provided

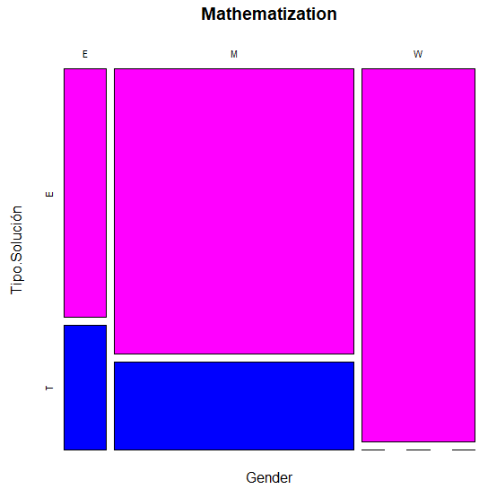

Figure 11

Discussion

In previous studies (Bolaños Evia, G. R., et al., 2023), we found differences in the form students approached problem-solving across academic programs. This difference is not observed in this case, which may be because this scenario poses more specific questions, thereby guiding the problem-solving approach more directly. Although there is no statistically significant difference in problem-solving approaches based on the gender composition of the teams ($p = 0.25$), all teams with a female majority provided explicit solutions. This result may be influenced by the fact that, in the Automotive Systems program, all teams were composed mostly of men. This reflects what naturally occurs in the academic program.

One objective of the study was to measure the level of vertical mathematization. However, due to the nature of the scenario and the time limit assigned for solving it (a maximum of 60 minutes), teams operated between the situational and referential levels. To reach a general or higher level, the scenario would need greater flexibility and complexity, encouraging students to propose original, intuitive solutions that they could then formalize using their mathematical knowledge or more advanced mathematics. In this scenario, we observe the interaction between horizontal and vertical mathematization. The highest level of horizontal mathematization reached was 6, which naturally places the vertical mathematical level at a referential level. To promote a higher level of mathematization in this scenario, requesting animations, programming, or alternative trajectories might be useful.

A particular issue detected in some cases was that the x-axis scale presented was inconsistent (not proportional to distance or disordered), which affected their progress in answering questions 4 and 5. We believe this is partly because, in general, when graphs are presented to students, the axis scales are predefined. However, when students face the problem of graphing functions or equations, they often do not understand the importance of establishing an appropriate scale. Even when using technological tools, they may misinterpret a graph due to the automatic scaling applied by the software.

Conclusions

As stated at the beginning, continuity comes naturally; most teams attempted to construct a continuous piecewise-defined function; however, most could not formulate it correctly. While some teams introduced their notations for piecewise functions, these were often imprecise or lacked a formal definition. The Folding Back process was observed during the scenario's solution, particularly between questions 3,4, and 5. Mathematization. Contrary to expectations, no differences were observed in horizontal mathematization between academic programs, semesters, type of argument, or use of roots at levels 3, 4, and 5. The difference is observed in horizontal mathematization between explicit and theoretical solutions, with explicit solutions demonstrating a higher level.

Recommendations

A possible next step in this line of research is to explore the impact of implementing broader, realistic problem-based learning scenarios that can be developed over extended periods, potentially outside the classroom, where use of technology is permitted and supports meaningful learning. Such an approach may enhance student engagement. To improve understanding of appropriate scaling in graphical representations, a focus on scales and proportions in graphical representations, incorporating activities that challenge students to define scales manually before using software tools, is recommended. Future studies could explore the relationship between team gender composition and the types of solutions proposed, considering more diverse and balanced samples across academic programs.

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