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Geometrical Optimization in the SoftAge

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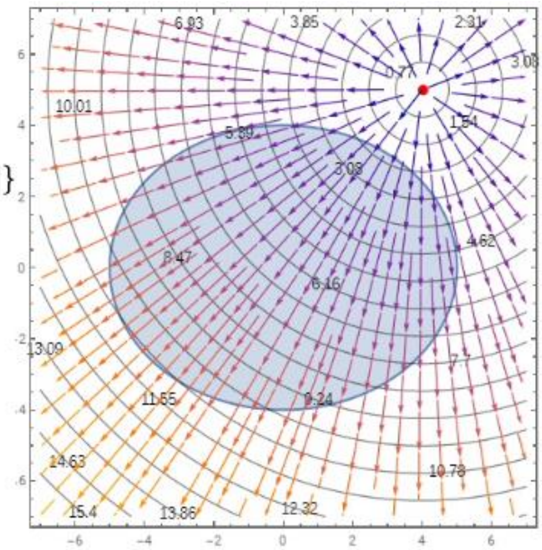
Abstract: We are increasingly making use of software in such a way that we should review education in mathematics. The concepts of level curves and gradient field empowered by software images help the understanding of strategies and resolution of optimization problems by the mathematics user in the softage. More than obtaining solutions from straight commands and with the help of the geometric approach the mathematics user can realize the situations when these commands help or how to deal with a problem whenever they are not so helpful.

Keywords: Optimization, Level Curves, Gradient Vector Field, Software.

Introduction

Let's consider the problem of finding the shortest and the longest distances from the point {4, 5} to the elliptic region $\left(\frac{x}{5}\right)^2+\left(\frac{y}{4}\right)^2 \leq 1$. It is useful to enter the following commands:

```
h[x_, y_] = (x/5)^2 + (y/4)^2;
a = RegionPlot[h[x, y] ≤ 1, {x, −7, 7}, {y, −7, 7}];
b = Graphics[{PointSize[Large] , Red , Point[{4, 5}]}
g[x_, y_] = (x − 4)^2 + (y − 5)^2;
c = ContourPlot[√g[x, y],
{x, −7, 7}, {y, −7, 7}, ContourShading → False,
ContourLabels → True, Contours → 20];
gradg[x_, y_] = Grad[g[x, y], {x, y}];
d = StreamPlot[gradg[x, y], {x, −7, 7}, {y, −7, 7}];
Show[a , b , c, d]
```



The picture shows that the shortest and the longest distances occur between boundary points of the elliptic region and the fixed point {4,5}. We can obtain these points by using Lagrange equations, which assert the tangency as a proportionality between gradients.

```
gradh[x_, y_] = Grad[h[x, y], {x, y}];
NSolve[{gradg[x, y] == λ gradh[x, y], h[x, y] == 1}, {x, y, λ}]
{{x → −4.03116, y → −2.36639, λ → 49.8067},
{x → 11.6477 − 11.0981 i, y → −9.31816 − 8.87852 i, λ → 20.5 − 4.28769i},
{x → 11.6477 + 11.0981 i, y → −9.31816 + 8.87852 i, λ → 20.5 + 4.28769 i},
{x → 2.95799, y → 3.22493, λ → −8.80673}}
```

∇h points towards the region outside which aligns and is opposite to ∇h in the maximum and minimum points which are:

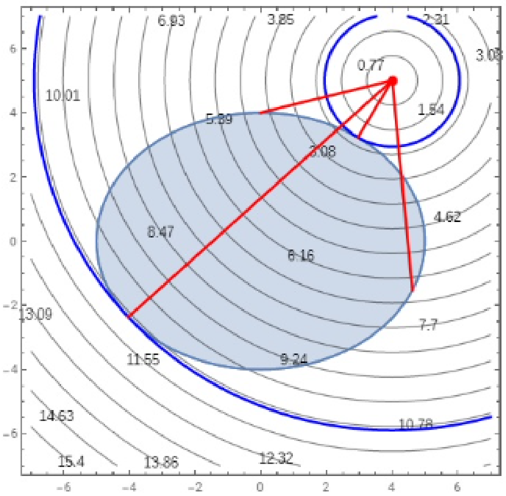
```
{Sqrt[g[2.95799, 3.22493]], Sqrt[g[−4.03116, −2.36639]]}
{2.05831, 10.8979}
```

The only point in which ∇g vanishes is {4,5} which is exterior to the region. Well, we could have parametrized the region boundary by $t \to \alpha(t)$ and on these points ∇g and α' are orthogonal, once:

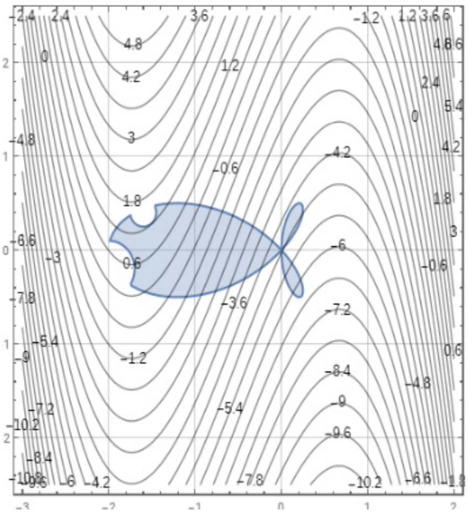
$$\frac{\mathrm{d}}{\mathrm{d}t} g(\alpha(t)) \;=\; \nabla g(\alpha(t)) \bullet \alpha'(t) \;=\; 0,$$

that explains the proportionality between ∇g and ∇h.

```
Solve[And[D[g[5 Cos[t], 4 Sin[t]], t] == 0, 0 ≤ t ≤ 2π, t];
{t[1], t[2], t[3], t[4]} = {%[[1, 1, 2]], %[[2, 1, 2]],
π/2, 15π/8};
N[Table[Sqrt[g[5 Cos[t[j]], 4 Sin[t[j]]]], {j, 1, 2}]]
ContourPlot[Sqrt[g[x, y]] == %, {x, −7, 7},
{y, −7, 7}, ContourStyle → {Blue,Thick}];
Show[a, b, c, e, %, Graphics[{PointSize[Large],
Red, Thick, Line[Table[{{4, 5}, α[t[j]]}, {j, 1, 4}]],
Point[Table[α[t[j]], {t, 1, 4}]]}]]
```



A little more complicated problem is the one to find the maximum and minimum points of $g(x, y) = 1.8y + (x + 1.2)(x + 2.2)(x - 1.8)$ in a region inspired by one of Kepler's Folium (Gray) cases given by $(x^2 + y^2)(x(x+2) + y^2) \leq 4x\,y^2$, $(x + y)^2 + (y + 0.2)^2 \geq (0.30)^2$ and $(x + 1.6)^2 + (y - 0.4)^2 \geq (0.15)^2$. The region is compact and since ∇g never vanishes, we know that we find maximum and minimum points in the region boundary.



$g[x\_, y\_] = 1.8y + (x+1.2)(x+2.2)(x-1.8);$

$\text{waves} = \text{ContourPlot}[g[x,y], \{x,-3,2\}, \{y,-2.5,2.5\},$
$\text{ContourShading} \to \text{False}, \text{ContourLabels} \to \text{True},$
$\text{Contours} \to 30];$

$\{h1[x\_, y\_], h2[x\_, y\_]\} = \{(x+2)^2 + (y+0.2)^2,$
$(x+1.6)^2 + (y-0.4)^2\};$

$\text{keplerfish} = \text{RegionPlot}[\text{And}[(x^2+y^2)(x(x+2)+y^2) \leq 4xy^2,$
$h2[x,y] \geq (0.15)^2, h1[x,y] \geq (0.30)^2],$
$\{x,-3,2\}, \{y,-2.5,2.5\}, \text{GridLines} \to \text{Automatic},$
$\text{PlotPoints} \to 100];$

$\text{Show}[\text{peixe}, \text{ondas}]$

Except for the two circumference arcs, this boundary is Kepler’s Folium, which can be written in polar coordinates as r(θ) = cosθ(4sinθ² - 2), θ ∈ [0, π] and therefore parametrized by α(t) = r(t){cos t, sin t}.

$r[t\_] = \text{Cos}[t]\ (4(\text{Sin}[t])^2 - 2)$

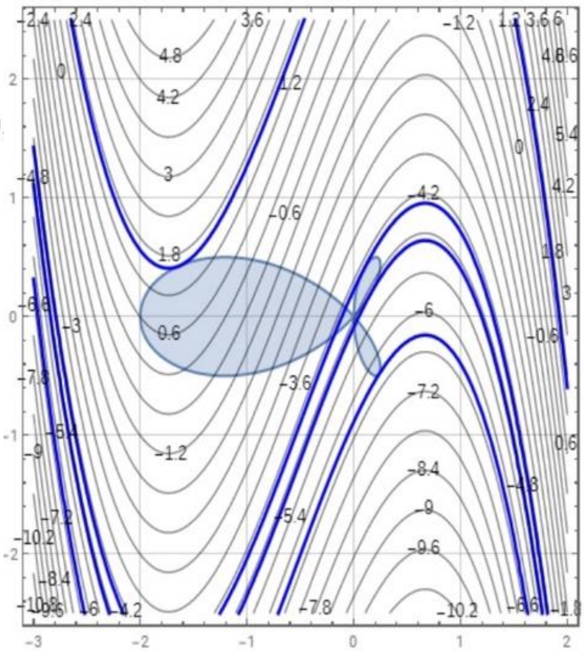
$\tan = \text{NSolve}[\text{And}[\text{D}[g[r[t]\ \text{Cos}[t], r[t]\ \text{Sin}[t]], t] == 0$
$0 \leq t \leq \pi]], t];$

$\text{gvec}[t\_] = g[r[t]\ \text{Cos}[t], r[t]\ \text{Sin}[t]];$

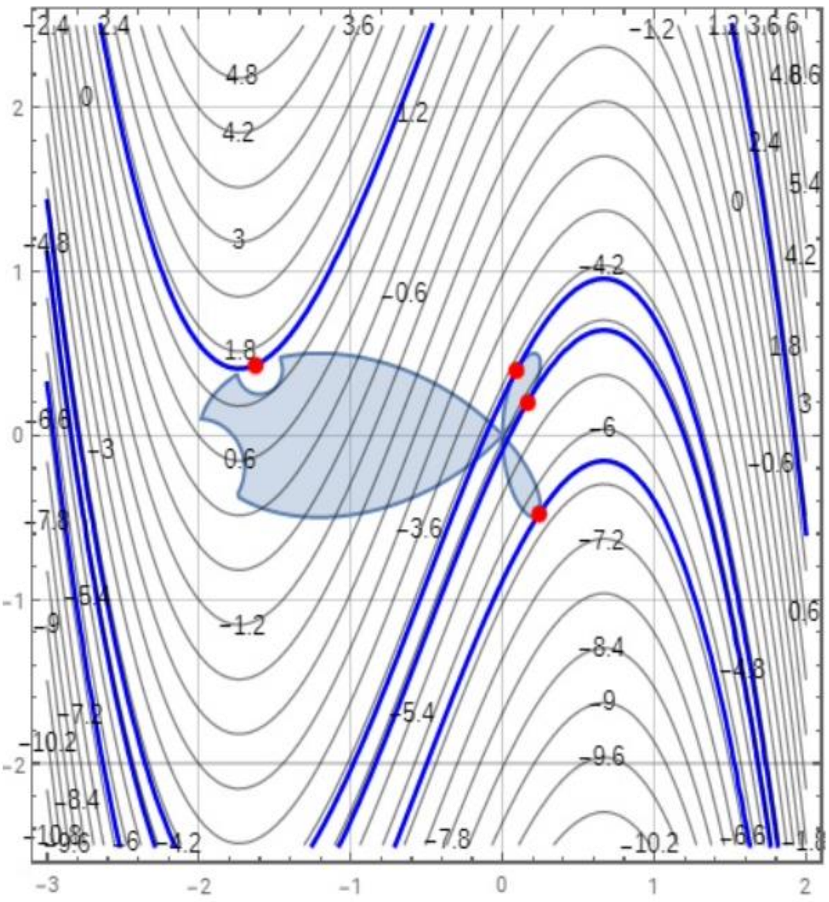
$\text{Table}[\phi[j] = \tan[[j,1,2]], \{j,1,4\}];$

$\text{gvec}[\%]$

$\{-4.91118, -4.34535, -6.34595, 1.61074\}$



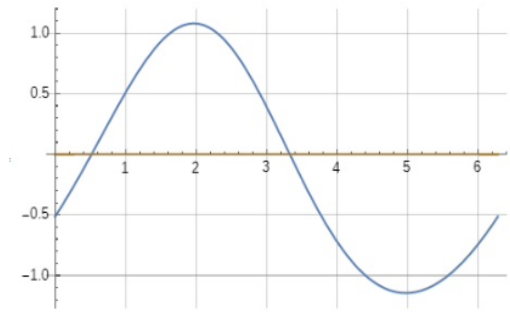
Well, this case of Kepler's folium is not the boundary of our region, but it is only its inspiring muse. We have found four points in which there is tangency between some level set of *g* and the folium. From these points, only those which also are points in the boundary of the region we have considered will be the candidates to be maximum or minimum points.



We have found the minimum of g in the region, $\cong 4.92$ as the picture shows. It also shows that the minimum occurs in one of the two points that connect Kepler's fish eye to Kepler's folium.

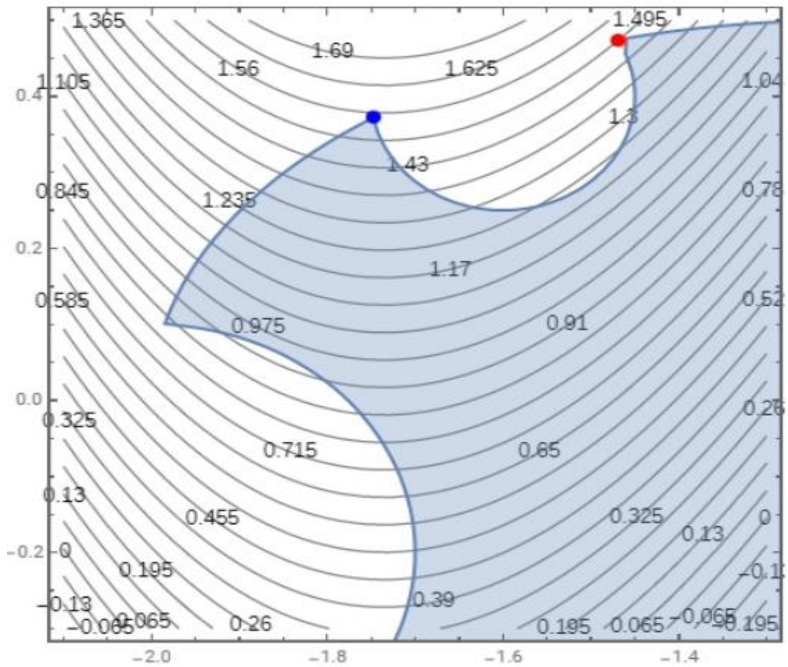
```
h[t_] = ((x^2 + y^2)(x(x + 2) + y^2) − 4xy^2)/ • {x → −1.6 + (0.15) Cos[t], y → 0.4 + (0.15) Sin[t];
Plot[{h[t], 0}, {t, 0, 2π}, GridLines → Automatic]
```

```
{FindRoot[h[t], {t, 0.5}], FindRoot[h[t], {t, 3.3}];
{t1, t2} = {%[[1, 1, 2]], %[[2, 1, 2]]};
β[t_] = {−1.6 + (0.15) Cos[t], 0.4 + (0.15) Sin[t]};
{p1, p2} = {β[t1], β[t2]};
{g[p1[[1]], p1[[2]]], g[p2[[1]], p2[[2]]]}
{1.49767, 1.54849}
p2
{−1.74736, 0.372006}
```



We have used Newton's method to find the roots and we can see that the maximum of $g \cong 1.55$ is in the left corner of Kepler's fish eye.

```
newwaves = ContourPlot[g[x, y], {x, −2.1, −1.3}, {y, −0.3, 0.5},
ContourShading → False, ContourLabels → True, Contours → 30];
Show[newondas, keplerfish, Graphics[{{PointSize[Large], Red, Thick,
Point[p1]}, {PointSize[Large], Blue, Thick, Point[p2]}}]]
```



Recommendations

We recommend this type of work for engineering and math students, who need to learn calculus with applications and analyze real optimization situations, review math education and general engineering education. Calculation optimization activities performed with software and geometric analysis tend only to bring learning gains. We must make this kind of approach more and more natural and every day.

Acknowledgements

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References

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