

An Evaluation of Mathematical Models and Stability Analysis of Learning Based on Reaction Kinetics

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Abstract: Human cognition and consciousness are perhaps the most confounding mystery. Somehow it has a linkage to the process of learning and storage of short-term and long-term memory in the form of knowledge. This paper examines a brief background of early models in learning presented by Atkinson and Shiffrin (1965) and related stochastic models utilizing probability functions. Each of these learning models capture certain facets of the learning process but are ineffective in describing the physical basis in which learning occurs. For this reason, this paper explores analogous mathematical models based on reaction kinetics that have been shown to represent chemical reactions found in nature. Six learning models are presented of unitary, binary, reversible binary, reversible binary with mass action, and enzyme learning model reactions with and without decay. Preliminary analysis of time series plots, phase line diagrams, and phase plane plots were conducted to illustrate equilibrium conditions and stability of the models. Each model is examined in terms of its limitations in the philosophy and inability to capture certain elements that are understood about the learning process. Finally, this paper concludes that the feasibility of understanding behavior such as stability through the tools of applied mathematics and thereby illuminating certain layers of human cognition and learning is a useful tool in examining the suitability of a possible deterministic model that could describe the learning process. Further analysis with empirical data would validate the suitability of the presented models.

Keywords: Engineering education, Theories of learning, Applied mathematics, Project-based learning, Atkinson-Shiffrin Theory

Introduction

Of the many mysteries that remain, human cognition and consciousness is perhaps the most confounding one. Both are inextricably tied to a vast array of knowledge that is present in an individual. This knowledge, accumulated over the life of the individual, shapes her/his perception. In both professional and personal lives, decisions are based upon information at hand combined with past learning. In fact, every single advancement throughout the human history is largely a result of building upon the learning and discoveries that have been

made by others.

The objective of every educational institution is to transfer knowledge and information to students through the process of learning so that the acquired knowledge can be translated into decisions, skills, and actions as they enter the workplace. As such, learning as a process is of utmost importance but is largely not well understood. It has easily been established that learning requires concentration, time, focused intellectual work, as well as mental wellbeing. Numerous mathematical models have been created that attempt to capture the learning process as well as the simultaneous process of forgetting (Šimon & Bulko, 2014). The aim of this paper is to examine the feasibility of learning models developed based on reaction kinetics using applied mathematics tools. Describing learning as a reaction does mimic the experience of the learner when achieving an “Aha” moment in understanding and offers a potential philosophical explanation for many of various variables that are understood to influence learning such as time, quality of instruction, motivation, genetics, and background knowledge. By representing learning as a process like a chemical reaction, an appropriate model could be used to justify incorporating problem-based learning into teaching curriculum by establishing that this learning strategy can function to increase long-term learning and thereby decrease the rate of memory loss.

Background

The Atkinson-Shiffrin Theory

The Atkinson-Shiffrin (AS) theory of human memory is a relevant concept to learning. Developed in 1968, it came about in a period that is considered the ‘cognitive revolution’ due to the tremendous amount of research being performed in experimental psychology (Atkinson & Shriffrin, 1965; Malmberg et al, 2019). At the same time, there was a simultaneous interest in describing these findings related to learning and memory with the elegance of mathematical modeling. The premise of this learning model is related to the storage of knowledge in the form of memory. In essence, learning is only effective if the knowledge gained can be retained. AS theory provides a model of learning based upon three memory blocks: sensory registry (SR), short-term memory (STM), and long-term memory (LTM). This theory presents an information processing model similar to a computer with inputs, outputs and processes in between. Initially, sensory organs detect information which later enters the sensory memory. If the information is deemed noteworthy, it can enter short term memory and then be transferred to the long-term memory if the information is repeated. If repetition does not occur, the information is forgotten. Even though this model has obvious limitations, namely, it is possible to create long-term memory without repetition, it served as a starting point to build upon to develop and refine mathematical models to describe the learning process as it applies to an academic setting. It is a logical conclusion to prioritize learning to achieve long-term memory and thereby prepare students for their careers with a strong and wide foundation (Malmberg et al., 2019).

There have been several mathematical models that have been developed by researchers in psychology, science teaching, mathematics, and others with the goal of representing the learning process. However, those models are empirical models that were developed by fitting input variables to numerical results; therefore, they fail to explain physiological process that occurs when an individual learns. An advanced numerical model that can

represent the biochemical process is greatly needed to better understand and quantify how a unique individual learns. Shikaa and Ajai built upon a logistical model first presented by the AS theory by introducing the Hicklin's concept of dynamic equilibrium theory to represent the concept of mastery learning (Shikaa & Ajai, 2015; Hicklin, 1976). Bush and Mosteller proposed learning as a combination of a myriad of factors related to probability (2006) while Anderson recognized that there was a rate-based element of a potential learning model (1983). Many of these models of learning in turn motivated a response in criticism to the logic in the development (Burros, 1952; Preece, 1984; Fey, 1961) highlighting the difficulty in describing the learning process effectively. However, although an accurate model to describe learning is heretofore nonexistent, many researchers are still motivated that it is possible to develop one and have done so by building upon the concept of learning as a probability and relating it to biological processes or plateau phenomena (Burini et al, 2016; Ormazábal et al, 2021, Wu et al, 2020).

Fredrick and Walberg presented a review of measures of instructional time as a predictor of outcomes to illustrate the importance of learning as a function of time in the theory of educational productivity (1980); while Aldridge presented a model of learning as a linear combination of independent variables (1983) using regression techniques and data based on the belief that students' performance is normally distributed. Aldridge's model was revised until it reached a form of a logistical model that is inherently stochastic. An empirical study that further inspected Aldridge's model attested that Aldridge's model has strengths and weaknesses common to many of the previous models. All these models contain some elements of learning but none of them can completely capture the multi-dimensional process of learning due to a lack of understanding of the biochemical process of learning. They also rely heavily upon empirical data for calibration rather than constants that are inherently deterministic (Aldridge, 1985).

Reaction Kinetics

Mathematical models of reaction kinetics are a powerful mathematical application involving differential equations that describe the basis of chemical kinetics as well as virus pathways and epidemics. Taking to heart the cliché that 'life is chemistry,' it is relevant to examine if the mechanisms that describe molecular biology and biochemical kinetics through processes such as enzyme kinetics or nerve signal propagation are analogous to the process of learning (Logan, 2013). Because reaction kinetics models have been successfully developed to describe several incredibly important natural phenomena, the theory of reaction kinetics is a logical approach to describe yet another natural phenomenon, learning.

Project-Based Learning (PJBL)

PJBL was developed in 1965 by 5 faculty of Health Sciences doctors led by founding Dean John Evans of McMaster University (Servant-Miklos, 2019). It is a learning approach in which students solve problems in small groups under the supervision of a tutor (Schmidt, 1993). The PJBL process is driven by the student, facilitated by the tutor, and is based on an educational approach where the learning is driven by problems or can be thought of as "learning through application". In this approach, learners (students) are encouraged to pursue

knowledge by asking questions. PJBL has been regarded as a key strategy for creating independent thinkers and learners in the medical education community (Servant-Miklos, 2019; Schmidt, 1993, Graaff & Kolmos, 2003).

Following the implementation of PJBL in the education of medicine, it has since been expanded to other fields and is considered a solution to some of the issues facing today's education (Graaff & Kolmos, 2003). For example, faculty at Weber State University established a PJBL center to achieve a double mission of being an active community member and providing opportunities for engineering students to gain needed skills in problem solving and project management (Foss, 2021). It has been found that the PJBL learning approaches greatly facilitated the training in competencies related to interpersonal skills and technical aptitude, experience of solving real-world problems from an engineering perspective, and collaborative learning (Foss, 2020). Liu and coworkers successfully integrated the PJBL mode in his senior mechanical engineering classes by introducing more than 20 projects from industry sponsors, university research centers, and a state agency (Liu, 2017). It was found from Liu's practice that the implementation of PJBL in course curricula struck the balance between achieving desired student learning outcomes and creating opportunities for enriching the student's educational experience (Liu et al, 2011; 2017-2019). As such, by examining the success of PJBL, there is evidence that learning model presented by AS has an aspect of application in addition to sensory registry.

Learning Models Based Upon Reaction Kinetics

Learning as Unitary Reaction (LUR)

The simplest learning model is a unitary reaction model in which a person is exposed to information, X, and then after a certain time, some amount of learning has occurred where knowledge now exists in the individual as shown in Eqn. (1).



Utilizing concepts in reaction kinetics, the rate of learning is proportional to the amount of information presented, or $r = kX$ where k is the learning constant for the individual. This means that information is 'consumed' by the individual at the rate of $-kX$, as shown in Eqn. (2).

$$\frac{dX}{dt} = -kX \quad (2)$$

Here the “-” indicates that the information is consumed in a learning reaction.

A function for information consumption that directly leads to knowledge can be solved from Eqn. (2) as a first order ordinary differential equation shown in equation 3 where X_0 represents the initial conditions of knowledge and k represents the rate constant of information consumption.

$$X(t) = X_0 e^{-kt} \quad (3)$$

This function is useful to establish a simple learning models where memory is perfect and all information that is presented is converted into knowledge. The LUR model also offers some expected results such as the form of an exponential function which indicates the rate at which learning occurs is not constant. With an examination of an exponential function, it can be concluded that the transfer of information into knowledge happens faster in the early stages of learning and slows down as time passes. This is in contrast to some research that shows a linear relationship between knowledge gained through learning and time (Fredrick & Walberg, 1980), but could be explained with the appropriate time scale. For example, an exponential function of time can be approximated as linear when the change in time becomes close to zero. The model of LUR is illustrated in Fig. 1, in which the information is consumed as a negative exponential function of time and acquired knowledge is increased as a positive exponential function of the time.

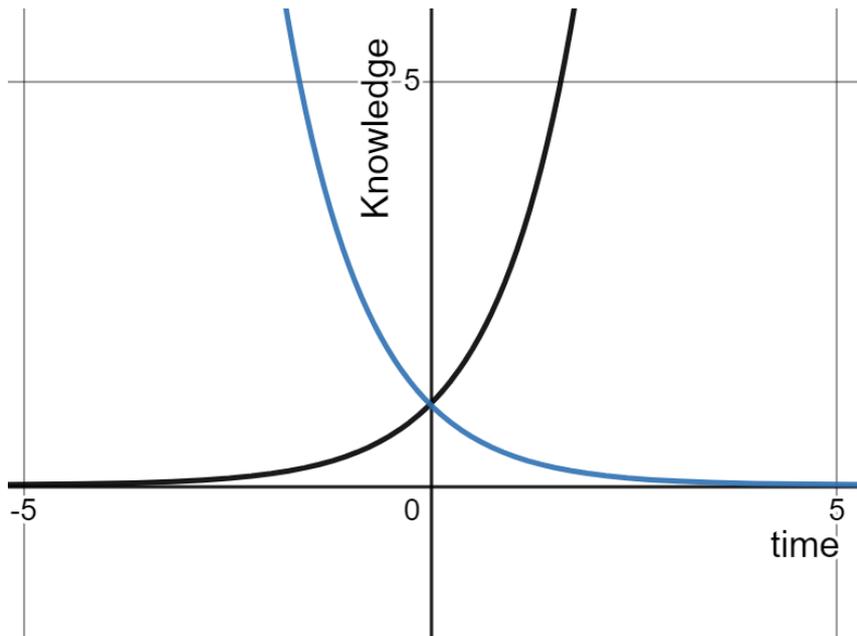


Figure 1. Learning as Unitary Reaction Time Series Plot

It should also be noted that the rate constant in many chemical reactions is not in fact constant and is instead a function of a number of variables. In the case of an individual's learning, the rate constant shown as k in the model could be understood as the Arrhenius rate constant shown in Eqn. (4).

$$k = Ae^{-\frac{E_a}{RT}} \quad (4)$$

Where A is the temperature independent pre-exponential factor, E_a is the activation energy for the reaction, R is the universal gas constant and T is the absolute temperature. By incorporating the Arrhenius rate constant into the LUR model, equation 5 is created.

$$X(t) = X_0 e^{-Ae^{-\frac{E_a}{RT}}t} \quad (5)$$

By expanding the LUR with the Arrhenius rate constant, certain elements that are understood about learning could be captured. The rate of learning is a function of quantities such as motivation which might include incentive or disincentive as well as an individual's capacity to learn coupled with their background knowledge and the quality of the learning environment. While likely a learning constant would include different variable than the Arrhenius rate constant, there may be an analogous form that is similar.

While this model is simple and easy to understand, it overlooks several significant features in a learning process. This model assumes that all information presented is consumed at the same rate and translated into knowledge and the reactions happens until all information is consumed. For example, a student may learn one topic at a different rate than another. The LUR model also neglects the memory loss, the counterpart of a reversible reaction in a chemical reaction. As such, the unitary reaction model must be expanded to consider learning as a binary reaction.

Learning as Binary Reaction (LBR)

In a binary chemical reaction, two different molecules combine to form a new product (Eqn. (6)):



Taking advantage of the AS theory, Eqn. (6) can be modified as a binary learning reaction model where Z is the gained knowledge; X represents the information that is gained through the senses and would include everything acquired from listening to a lecture to reading a text book or watching a video; and Y represents the interaction of the learner with the information and would include everything from solving a problem to completing a project following the PJBL methodology. The LBR model by nature, remains flawed because the model presumes that no knowledge can be gained from sensing alone and application of that information (practice) is necessary to be converted into knowledge. This is clearly not the case; however, this model likely does apply to certain fields. For example, in the practice of medicine, one may read volumes of books on surgery and observe others performing surgeries but likely will find that the extent of their knowledge gained from these sensing methods is insufficient to the practice of performing a surgery. For the quantity of knowledge to reach levels of mastery where one can become a professional surgeon, extensive application (practice) is necessary.

By applying the law of mass action, it can be established that the rate of the reaction, or learning, is proportional to the product of the two reactants X and Y and the rate constant k shown in Eqn. (7) (Logan, 2013):

$$r = kXY \quad (7)$$

The differential equations can be written by the following shown in Eqn. (8).

$$\frac{dX}{dt} = -r \quad \frac{dY}{dt} = -r \quad \frac{dZ}{dt} = r \quad (8)$$

From here, several conservation laws can be developed shown in Eqn. (9).

$$\frac{dX}{dt} - \frac{dY}{dt} = 0 \quad \frac{dX}{dt} + \frac{dZ}{dt} = 0 \quad \frac{dY}{dt} + \frac{dZ}{dt} = 0 \quad (9)$$

Which follows that $X - Y = \text{Constant}$, c , which can be used to rewrite the differential equations with one unknown in Eqn. (10) and (11).

$$\frac{dX}{dt} = -kXY = -kX(X-c) \quad (10)$$

$$\frac{dZ}{dt} = kXY = kX(X-c) \quad (11)$$

Which can be rewritten in the form of Eqn. (12) to resemble a logistic model which is consistent with many of the prior mathematical models on learning that have been presented.

$$\frac{dX}{dt} = kcX\left(1 - \frac{X}{c}\right) \quad (12)$$

where c is the learning capacity of an individual and the term kc can be represented as the intrinsic learning rate. Using Eqn. (12) to solve for $dX/dt = 0$, the stability of the model can be seen in the generic phase line plot Fig. 2. The condition of zero information is unstable and the condition when the information is equal to the learning capacity ($dX/dt = 0$) is stable. The LBR model offers some advantages over the LUR model as this model resembles a logistic function. In the LBR model, the constant c represents the ‘carrying capacity’ or plateau phenomena that can be used to describe learning as a function that represents diminishing returns with time as has been demonstrated by other presented models (Buirini et al, 2016; Ormazábal et al, 2021; Wu et al, 2020).

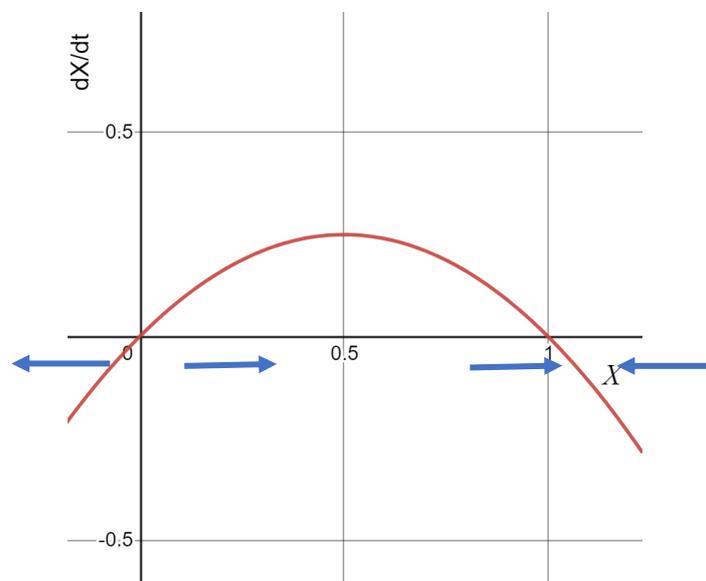
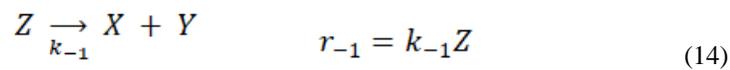
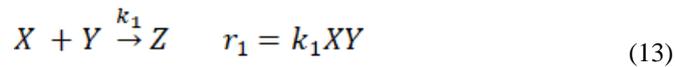


Figure 2. Generic Phase Line Plot of Binary Learning for $\frac{dX}{dt}$

There are, however, obvious limitations to the application of the LBR model. Just like the LUR model, it does not include a mechanism to capture the memory loss. It should also be noted that this model assumes a perfect translation of information into knowledge without any errors in conceptual understanding, which is counter to intuitive understanding of the learning process. While it does capture more that is understood about learning than the LUR model by including plateau phenomena, it needs to include a mechanism of memory loss through forgetting that can be captured as a reversible reaction.

Learning as Reversible Binary Reaction (LRBR)

One possible way to capture the memory loss associated with learning is to consider a reversible binary reaction. Using the chemistry analogy, an initial reaction of constituents X and Y forms the product Z with the rate constant k_1 as shown in Eqn. (13). The product, is broken down into its reactant constituents by a reversible reaction with rate constant k_{-1} as shown in Eqn. (14). Utilizing the law of mass action, the rate of reaction is proportional to the product of the reactant concentrations as shown by Logan (Logan, 2013) thereby allowing for an expression of r_1 and r_{-1} as shown in Eqns. (13, 14). Undoubtedly, this is not the real mechanism of the loss of knowledge but it is viable to use this mechanism to model the process of knowledge loss. In the LRBR model, a term k_{-1} is introduced, which represents the loss of knowledge or rate constant for the reversible reaction and would only be considered as a constant under certain situations. For example, if an individual is well-rested and able to focus on the information it could be expected that this constant would be comparatively smaller than if the individual is attempting to multi-task or ‘cramming’ in the wee hours of the evening.



With this model, the rate equations are shown in Eqn. 15 by utilizing the property that rates add for a system of reactions (Logan, 2013).

$$\frac{dX}{dt} = -r_1 + r_{-1}, \quad \frac{dY}{dt} = -r_1 + r_{-1}, \quad \frac{dZ}{dt} = r_1 - r_{-1} \quad (15)$$

From which the following conservation laws can be determined as shown in Eqn. 16.

$$\frac{dX}{dt} - \frac{dY}{dt} = 0 \quad (16)$$

Solving the differential equation shown in Eqn. (16) follows that $X - Y = \text{constant} = c$, an expression called a conservation law (Logan, 2013) is created. Eqn. (17) is the conservation laws for the remaining terms.

$$X - Y = c_1, \quad X + Z = c_2, \quad Y + Z = c_3 \quad (17)$$

These constants add new dimensions of learning because they are based upon initial conditions shown in Eqn. (18) at time $t = 0$. These initial conditions can be used to represent the background knowledge of an individual and the level of understanding that is present in foundational topics. They also can function as a way of capturing the attitude of the learner and the learner's preferences, strengths and weaknesses in learning new things. These psychological factors are complicated and numerous but would build into a framework that could be captured with initial conditions. For example, one qualitative factor that contributes to student success is the feeling of belonging. If, for example, a student feels that they do not belong in the field of study, this could influence their ability to learn the information presented and translate it into knowledge (Inzlicht & Ben-Zeev, 2000; Schmader & Johns, 2003, Rainey & Dancy, 2018, Inzlicht & Good, 2006). The combination of the constants derived from initial condition and rate constants is an innovative way of addressing the nature versus nurture dilemma in human behavior and development. Clearly there is a genetic component of learning captured by the k constants and an environmental or cultural component captured by the c constants.

$$X(0) = X_0 \quad Y(0) = Y_0 \quad Z(0) = Z_0 \tag{18}$$

The constants can be determined from the initial conditions shown in Eqn. (19).

$$X - Y = c_1 = X(0) - Y(0) \tag{19}$$

The above conservation laws shown in Eqn. (17) can be used to derive a differential equation for X and Z from Eqn. (15):

$$\frac{dX}{dt} = -k_1XY + k_1Z = -k_1X(X - c_1) + k_1(c_2 - X) \tag{20}$$

$$\frac{dZ}{dt} = k_1XY - k_1Z = -k_1(c_2 - Z)(c_3 - Z) + k_1Z \tag{21}$$

Analysis of Eqns. (20) and (21) have been done that illustrate the behavior of the LRBR model as shown in the generic phase line diagrams in Figs. 3-5. Figs. 3 and 4 show the phase line diagrams and stability for $\frac{dX}{dt}$ and $\frac{dZ}{dt}$ with positive c constants and Fig. 5 shows $\frac{dZ}{dt}$ with negative c constants. Figs. 4 and 5 can be used to understand how individuals with the same genetic makeup including capacity to learn and rate of learning could achieve knowledge in very different ways depending upon their initial conditions.

Like the LBR model, as illustrated in Figs. 3 and 4, two equilibrium conditions exist in the LRBR model (Eqn. (20 and 21)). Of particular note is a stability condition in which the information content is equal to the capacity of the individual. It also can be observed that the conditions where the change in knowledge is positive or negative, implying a theoretical ability to optimize the rate of change of knowledge.

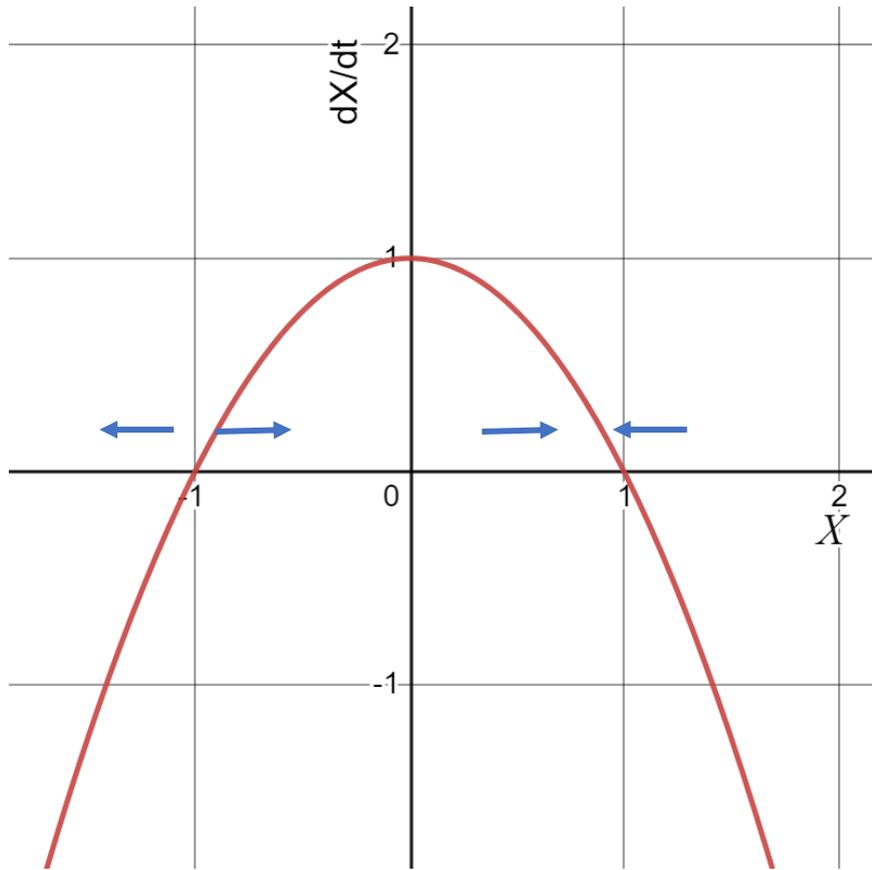


Figure 3. Generic Phase Line Diagram of Reversible Binary Learning for $\frac{dX}{dt}$ with Positive c Constants

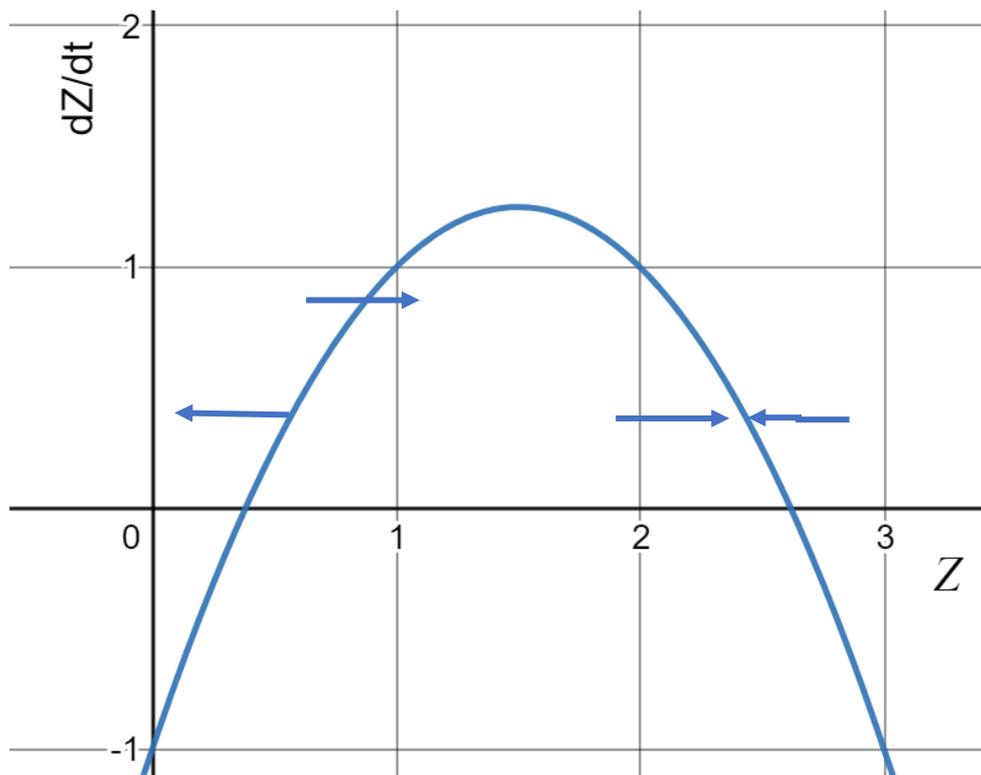


Figure 4. Generic Phase Line Diagram of LRBR for $\frac{dZ}{dt}$ with Positive c Constants

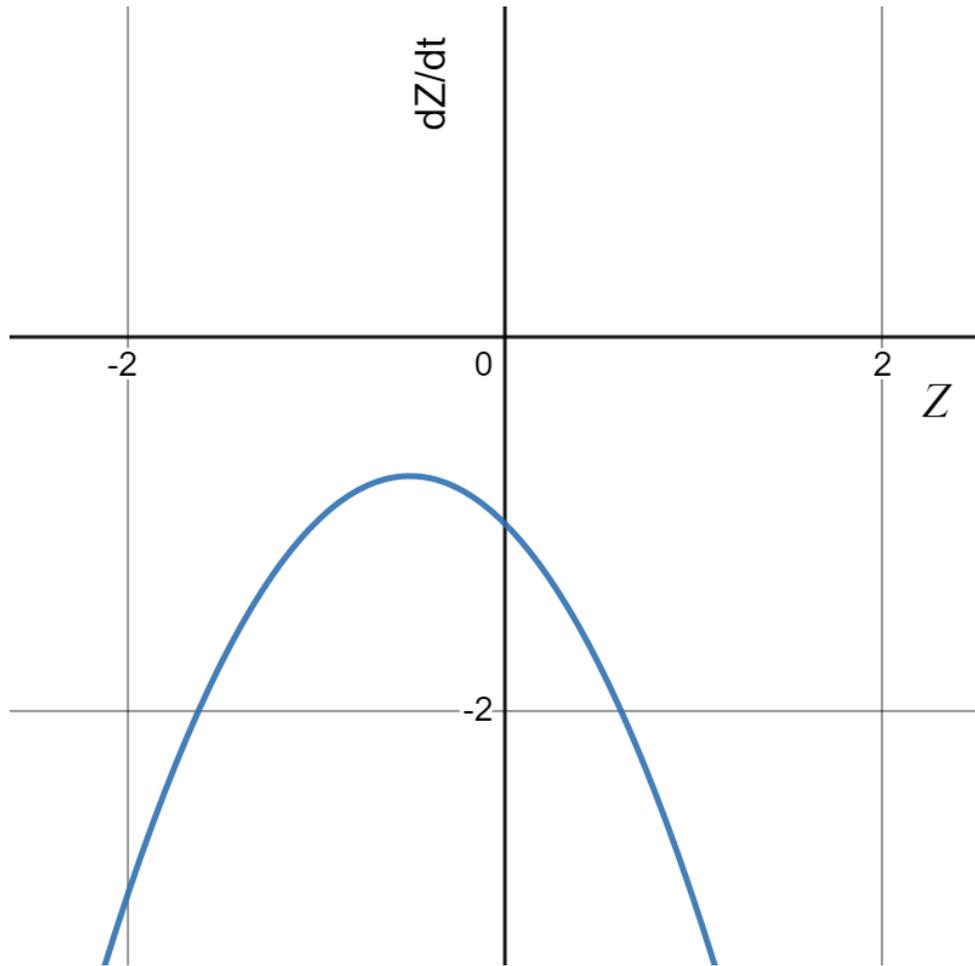


Figure 5. Generic Phase Line Diagram of LRBR for $\frac{dZ}{dt}$ with Negative c Constants

It is also of interest to examine the equilibrium conditions of this model. At equilibrium or $\frac{dZ}{dt} = 0$, an individual would not be retaining any knowledge from their efforts in learning. There are likely many students throughout history that have experienced equilibrium conditions in their attempt to learn new knowledge. When the differential equation $\frac{dX}{dt}$, Eqn. (20) is equal to zero Eqn. (22) is given.

$$k_1XY = k_{-1}Z \quad (22)$$

There are still several limitations that remain in this model and analysis. Namely, this model does not provide a way to identify short-term versus long-term knowledge. Also, this model cannot distinguish different methods of presenting the information; nor is the model able to differentiate effectiveness of differing methods of teaching. Thus, the LRBR model needs to be further improved to address these shortcomings.

Learning as Reversible Binary Mass Action Reaction (LMAR)

Building upon the limitations of the LRBR model, it is possible to continue with the reaction kinetics theory to

model knowledge loss through the unintentional act of forgetting as well as the formation of both short-term and long-term memory where the knowledge is stored. These two ‘reactants’ form the reversible reaction where both short-term and long-term knowledge are represented by W and Z, respectively (Eqn. (23)).



In above equation, m, n, p, and q are coefficients for each term, in which m and n are related to the quantity of the stimulus of information and p and q denote a theoretical quantity of short term and long term knowledge that has been gained by the rate and reversible reaction constants k_1 and k_{-1} . Utilizing the law of mass action, the rate of reaction is proportional to $X^m Y^n$ (Logan, 2013), the following rates can be defined as shown in Eqn. (24):

$$r_1 = k_1 X^m Y^n, \quad r_{-1} = k_{-1} W^p Z^q \quad (24)$$

From Eqn. (23), the rate equations are developed for each component, X, Y, W, and Z.

$$\frac{dX}{dt} = -r_1 + r_{-1} \quad \frac{dY}{dt} = -r_1 + r_{-1} \quad \frac{dW}{dt} = r_1 - r_{-1} \quad \frac{dZ}{dt} = r_1 - r_{-1} \quad (25)$$

The following differential equations (Eqns. (27-29)) can then be obtained by substituting Eqn. (26) into Eqn. (25) giving the following conservation laws where c_i s represent constants from initial conditions:

$$X - Y = c_1 \quad X + Z = c_2 \quad Y + Z = c_3 \quad W - Z = c_4 \quad X + W = c_5 \quad Y + W = c_6 \quad (26)$$

The differential equations then are obtained as shown in Eqn. (27-29):

$$\frac{dX}{dt} = -k_1 X^m Y^n + k_{-1} W^p Z^q = -k_1 X^m (X - c_1)^n + k_{-1} (c_5 - X)^p (c_2 - X)^q \quad (27)$$

$$\frac{dW}{dt} = k_1 X^m Y^n - k_{-1} W^p Z^q = -k_1 (c_5 - W)^m (c_6 - W)^n - k_{-1} W^p (W - c_4)^q \quad (28)$$

$$\frac{dZ}{dt} = k_1 X^m Y^n - k_{-1} W^p Z^q = k_1 (c_2 - Z)^m (c_3 - Z)^n - k_{-1} (c_4 + Z)^p Z^q \quad (29)$$

A generic phase diagram (Fig. 6) was created to better understand the behavior of the LMAR model with positive c constants and negative c constants (Fig. 7). In this model, it is assumed all constants are equal to 1 and all exponents are 2.

As can be seen in Fig. 6, this model presents one equilibrium condition that is stable. It also shows a very narrow range where the change of knowledge, $\frac{dZ}{dt}$ is positive and the total knowledge, Z is positive. Figure 7 presents one equilibrium condition that is unstable.

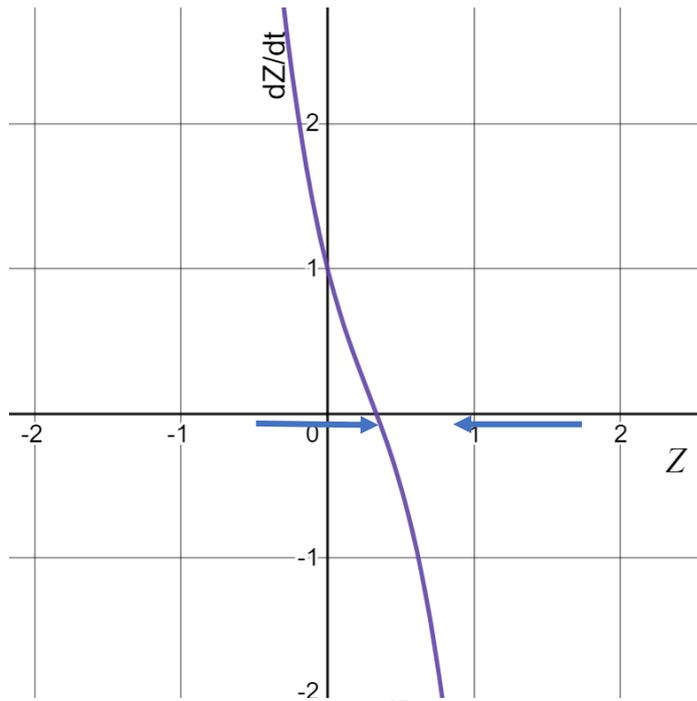


Figure 6. A Generic Phase Line Diagram of $\frac{dZ}{dt}$ for LMAR Model with Positive c Constants

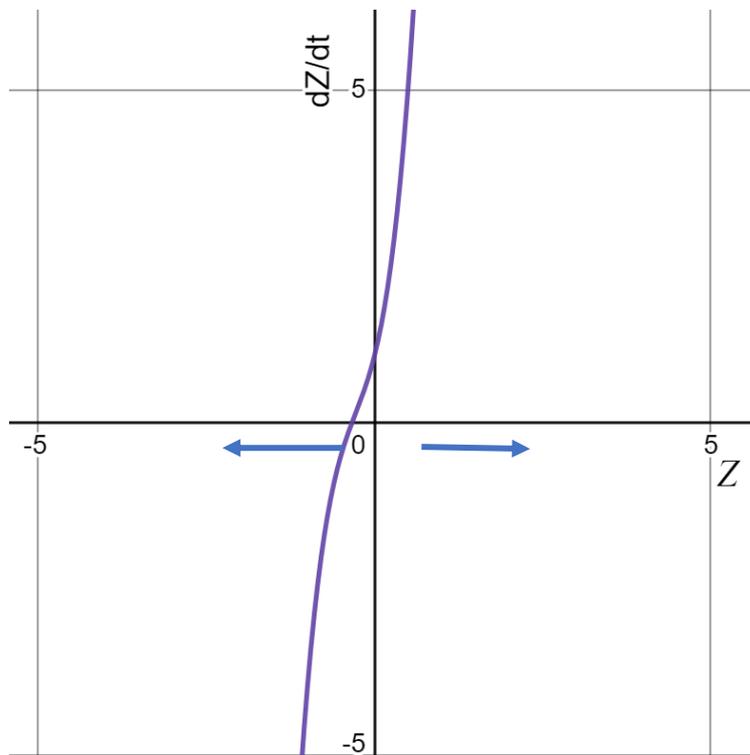


Figure 7. A Generic Phase Line Diagram of $\frac{dZ}{dt}$ for LMAR Model with Negative c Constants

There exist several flaws in the philosophy of this model. Firstly, this model presumes that the short-term and long-term knowledge are gained concurrently, which violates the AS theory. The model also includes many constants, whose values are very difficult to be calibrated from empirical data.

Learning as Enzyme Kinetic Reaction Model (LEKR)

In an enzyme kinetic model, an intermediate complex is created before the product is formed (Logan, 2013). This can be a useful analogous model to capture the change from information to the short-term memory represented by the variable W and transfer it through another process into the long-term memory represented by the variable Z . In this model, we assume that X represents the information that is to be consumed and Y represents the interaction with that information through an interaction like PJBL. In a typical reaction of a metabolic pathway catalyzed by an enzyme, the final product is produced and the enzyme is recovered (Logan, 2013). In the LEKR model, it makes more sense to model the catalyst, Y , as a consumable rather than recovered as shown in Eqn. (30).



Giving the following rates and rate equations shown in Eqn. (31, 32) from the law of mass action (Logan, 2013).

$$r_1 = k_1 X^m Y^n, \quad r_{-1} = k_{-1} W^p, \quad r_2 = k_2 W^p \quad (31)$$

$$\frac{dX}{dt} = -r_1 + r_{-1} \quad \frac{dY}{dt} = -r_1 + r_{-1} \quad \frac{dW}{dt} = r_1 - r_{-1} - r_2 \quad \frac{dZ}{dt} = r_2 \quad (32)$$

Giving the following conservation laws shown in Eqn. (33):

$$X - Y = c_1 \quad X + W + Z = c_2 \quad Y + W + Z = c_3 \quad (33)$$

The differential equations become:

$$\frac{dX}{dt} = -k_1 X^m Y^n + k_{-1} W^p = -k_1 X^m (X - c_1)^n + k_{-1} W^p \quad (34)$$

$$\frac{dW}{dt} = k_1 X^m Y^n - k_{-1} W^p - k_2 W^p = k_1 X^m (X - c_1)^n - k_{-1} W^p - k_2 W^p \quad (35)$$

$$\frac{dZ}{dt} = k_2 W^p \quad (36)$$

The nonlinear systems of Eqns. (34-36) can be examined further with phase plane plots. For simplicity, generic phase plane plots are created assuming all exponents, (m, n, p) are unitary values equal to 1 as well as constants, c_i . While this assumption does eliminate many of the complexities of a learning model that can capture not only the qualities of the instruction and genetic ability of the individual, the simplification does allow for the evaluation of stability through phase plane plots and as a result does illuminate certain elements of the learning process. Figs. 8, 9, and 10 illustrate the relationship between X and W , W and Z , and X and Z respectively.

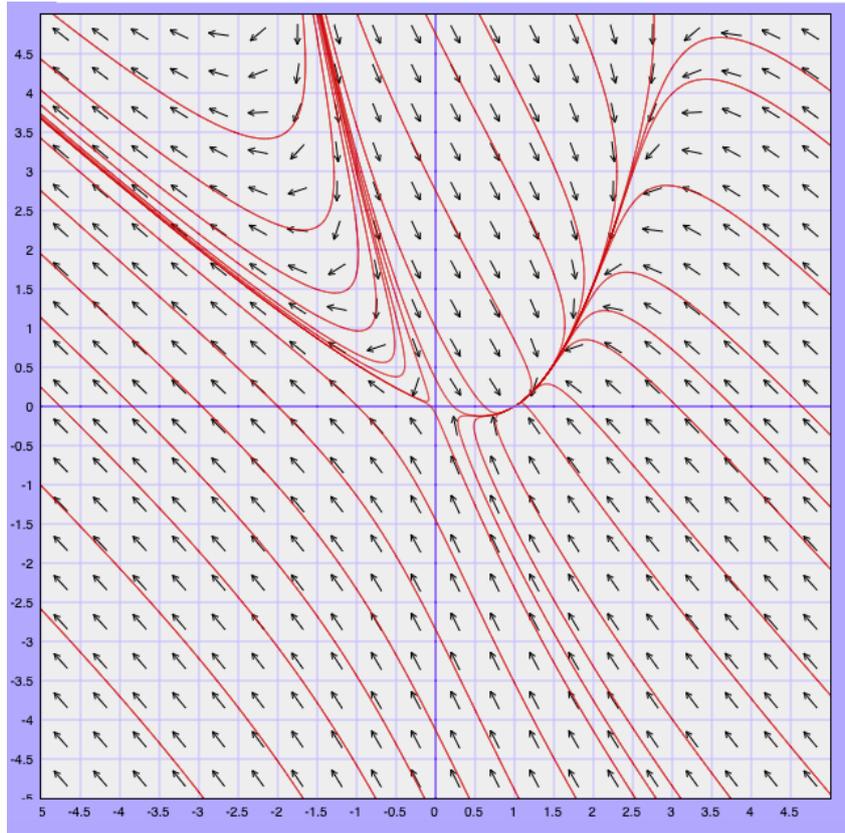


Figure 8. Generic Phase Plane Plot of X and W based on the LEKR Model

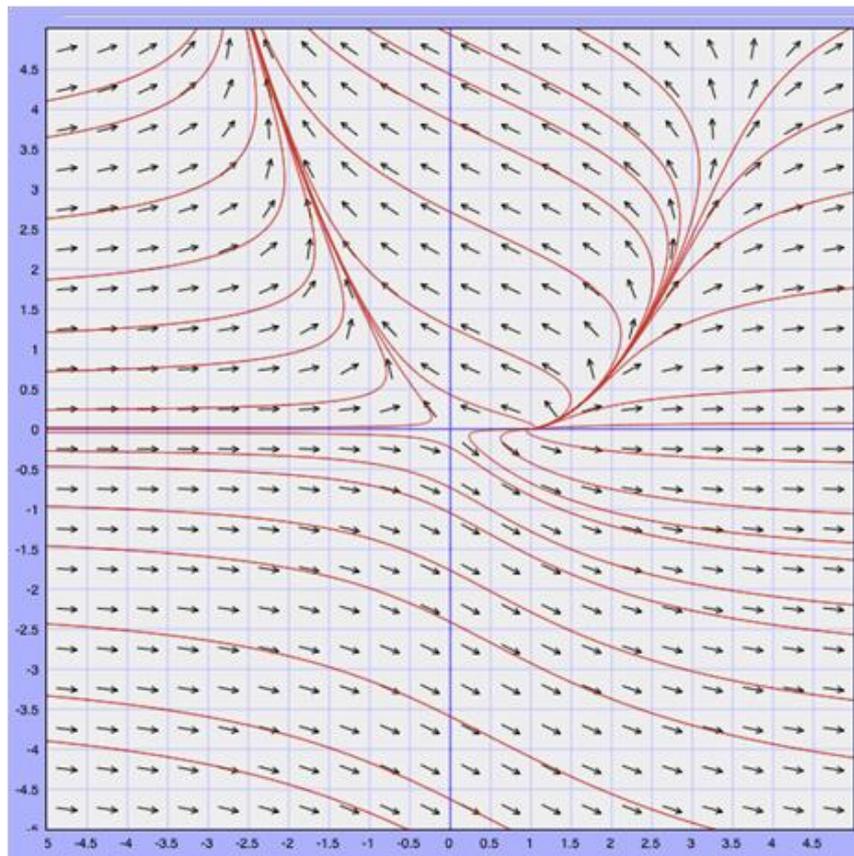


Figure 9. Generic Phase Plane Plot of W and Z based on the LEKR Model

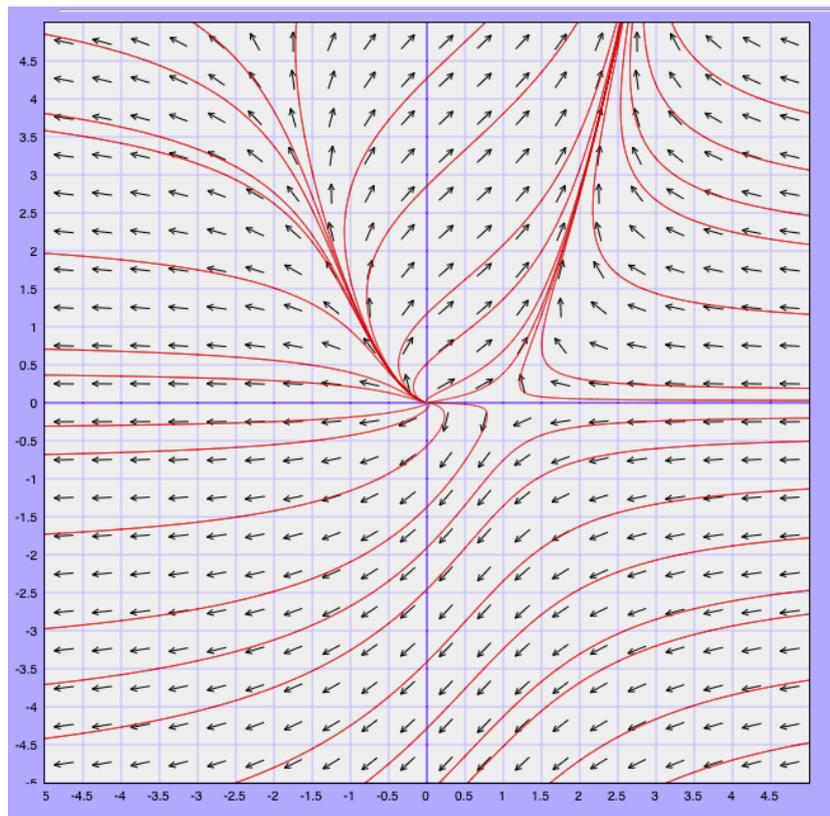


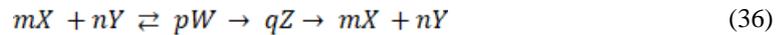
Figure 10. Generic Phase Plane Plot of X and Z based on the LEKR Model

The most interesting observation that are offered by Figures 8-10 is an examination of stability. When graphing parametric equations on the phase plane such as those in Eqn. (34-36), the equilibrium or steady state condition is represented by a critical point. By examining these diagrams from the basis of the Poincaré-Bendixson theorem, stability can also be observed which represents a degree of permanence in an equilibrium solution. A critical point is stable if all paths that are nearby remain near the point for all time $t > 0$ (Logan, 2013). Fig. 8 indicates a stable node between the sensory information, X and the short-term memory, W; however, there are no stability points between any functions of Z, or the long-term memory. This would be supported by a hypothesis that forgetting is a constant process and that no knowledge can be retained indefinitely. However, this concept would be refuted when examined alongside research in the genetic component of trauma as some recent studies have suggested traumatic experiences can alter genes and be passed down (Youssef et al, 2018). This research would support a model that does demonstrate stability as it would indicate that some learning can occur on a genetic level that is beyond the time scale of the individual. This might imply that there is some stability in the long-term memory, Z counter to this model.

Of course, this model still has its own limitations. For example, there is no process in this model that describes the decay of long-term memory nor is there a process in which the short-term memory is bypassed and the long-term memory is formed initially, both of which should be represented in a learning model. However, as each of the models discussed in this paper build in complexity, certain elements that align with observations of the learning process are developed and captured.

Learning as Enzyme Kinetic Reaction with Decay Model (LED)

Improving upon the LEKR model, a process to represent the decay of long-term knowledge is introduced in the learning as enzyme kinetic reaction with decay (LED) model. The process of decay, or reduction in the term, Z, represents long-term memory loss and is a necessary component to include in a learning model. Similar to the LEKR model, short-term memory is represented by the variable W and transfers through another process into the long-term memory represented by the variable Z. In this model, we assume that X represents the information that is to be consumed and Y represents the interaction with that information through an interaction like PJBL. In a typical reaction of a metabolic pathway catalyzed by an enzyme, the final product is produced and the enzyme is recovered (Logan, 2013) and the decay process generates the terms mX and nY. These terms, are not intended to be quantifiable or measurable but rather are defined by the rate of the increase or decrease in short term or long term knowledge. It is reasonable to presume that information is neither created nor destroyed nor consumed in the process of learning. The variables m, n, p, q, and r represent a theoretical constant that could be used to customize the model to genetic and environmental effects. The LED model is shown in Eqn. (36).



Giving the following rates and rate equations shown in Eqn. (37, 38) from the law of mass action (Logan, 2013).

$$r_1 = k_1 X^m Y^n \quad r_{-1} = k_{-1} W^p \quad r_2 = k_2 W^p \quad r_3 = k_3 Z^q \quad (37)$$

$$\frac{dX}{dt} = -r_1 + r_{-1} + r_3 \quad \frac{dY}{dt} = -r_1 + r_{-1} + r_3 \quad \frac{dW}{dt} = r_1 - r_{-1} - r_2 \quad \frac{dZ}{dt} = r_2 - r_3 \quad (38)$$

Giving the following conservation laws shown in Eqn. (39) which interestingly mimic the conservation laws in Eqn. (33):

$$X - Y = c_1 \quad X + W + Z = c_2 \quad Y + W + Z = c_3 \quad (39)$$

The differential equations become:

$$\frac{dX}{dt} = -k_1 X^m Y^n + k_{-1} W^p + k_3 Z^q = -k_1 X^m (X - c_1)^n + k_{-1} W^p + k_3 (c_2 - X - W)^q = -k_1 X^m (X - c_1)^n + k_{-1} (c_2 - X - Z)^p + k_3 Z^q \quad (40)$$

$$\frac{dW}{dt} = k_1 X^m Y^n - k_{-1} W^p - k_2 W^p = k_1 X^m (X - c_1)^n - k_{-1} W^p - k_2 W^p = k_1 (c_2 - W - Z)^m (c_3 - W - Z)^n - k_{-1} W^p - k_2 W^p \quad (41)$$

$$\frac{dZ}{dt} = k_2 W^p - k_3 Z^q = k_2 (c_2 - X - Z)^p - k_3 Z^q \quad (42)$$

The nonlinear systems of equations (40-42) can be examined further with phase plane plots. For simplicity,

generic phase plane plots are created assuming all exponents (m, n, p) are unitary values equal to 1 as well as constants, c_i as shown in Figures 11-13. While this assumption does eliminate many of the complexities of a learning model that can capture not only the qualities of the instruction and genetic ability of the individual, the simplification does allow for the evaluation of stability through phase plane plots and as a result does illuminate certain elements of the learning process.

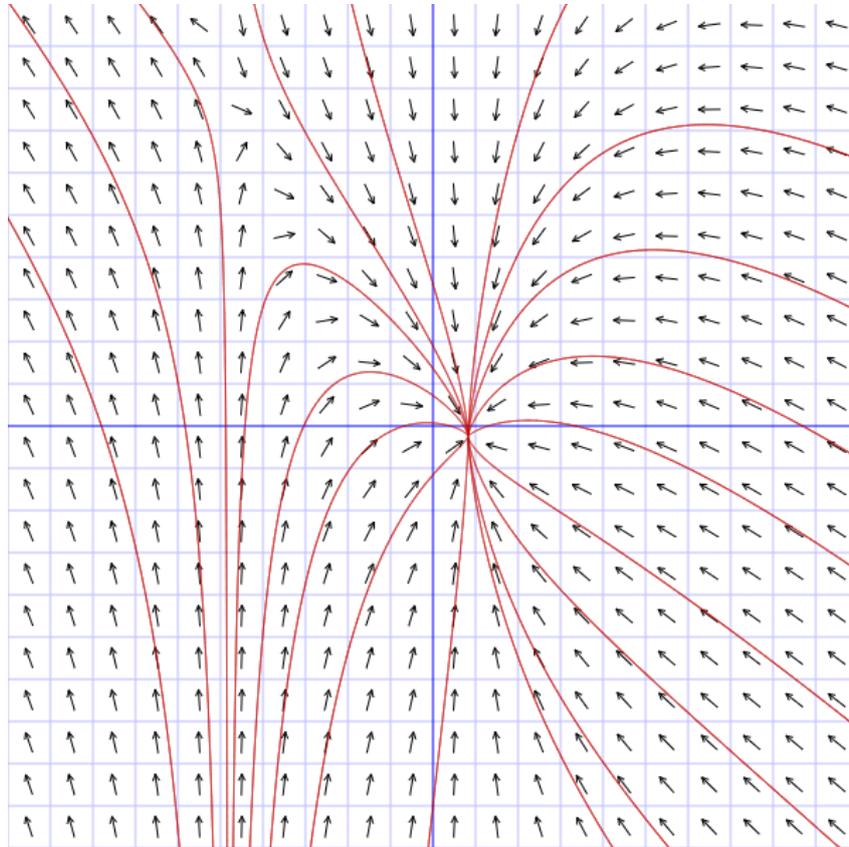


Figure 11. Generic Phase Plane Plot of X and W based on the LED Model

The most interesting observation that are offered by Figures 11-13 is an analysis of stability. The equilibrium or steady state condition is represented by a critical point and stability can also be observed which represents a degree of permanence in an equilibrium solution. Fig. 13 indicates a stable node between the sensory information, X, and the short-term memory, W, similar to the LEKR model.

Where the LED model diverges is obvious in the stability analysis of Figures 12 and 13. While the LEKR model did not offer any stable points of equilibrium, the LED model does clearly indicate the presence of a stable spiral in Figure 12 which indicate the presence of an imaginary component of the solution. This could be one possible explanation why thus far a learning model has not been developed as the examination of learning occurring on the imaginary plane has not been done. Most research has been limited to the Cartesian plane of analysis thus far. Figure 13 also indicates a stable node between X and Z.

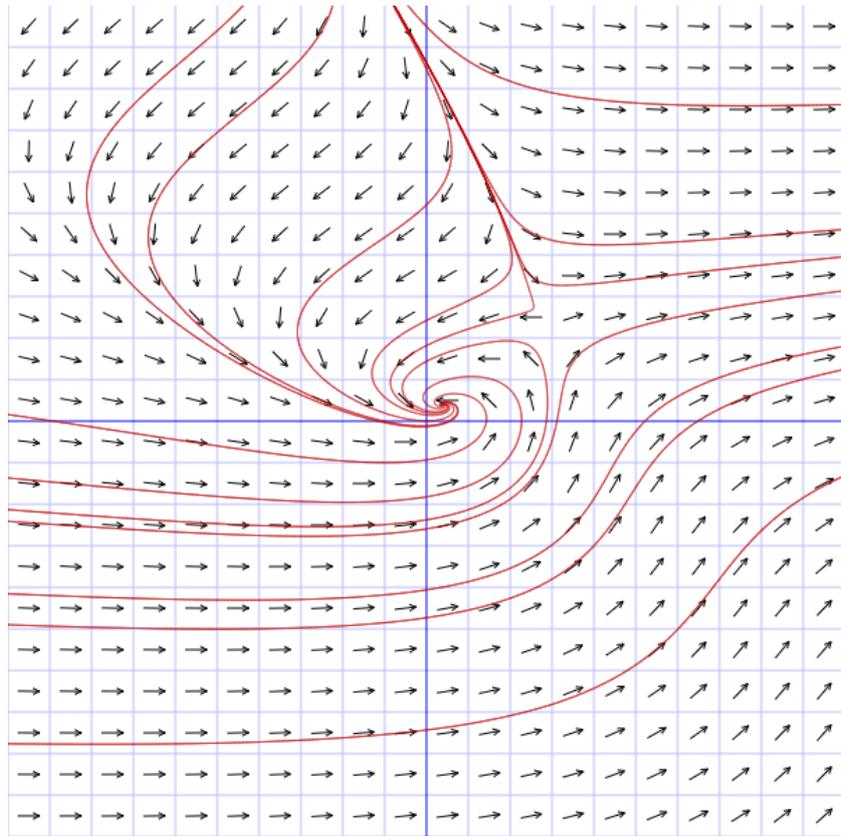


Figure 12. Generic Phase Plane Plot of W and Z based on the LED Model

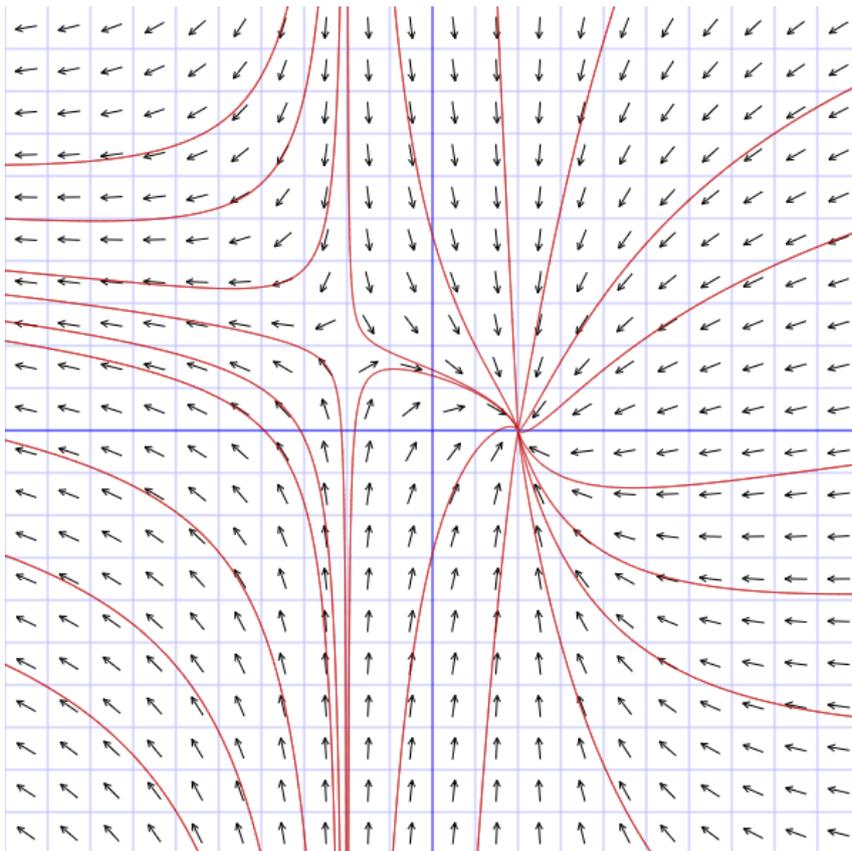


Figure 13. Generic Phase Plane Plot of X and Z based on the LED Model

Conclusion

Of the many mysteries that remain, human cognition and consciousness has likely been the most considered and studied yet remains one of the most mysterious of topics. Indeed, such a topic has been the focus of much research on the human brain including psychology and neuroscience and though significant advancements have occurred, conclusions regarding human consciousness are limited in scope. The goal of this paper is to use tools used in the field of applied mathematics to gain an understanding of behavior on six different theoretical learning models that are deterministic instead of stochastic in nature. It is the conclusion of the authors that perhaps we are not equipped with the suitable cognitive capacity to fully understand the mystery of life or do not yet have a thorough enough understanding on biological processes to create a complete mathematical model. As such, it is the goal of this paper to examine the possibility that the human experience could be captured in the language of nature; mathematics. It is possible that the experience of learning is analogous to the microscale processes happening within Chemistry and that learning is similar to a chemical reaction and can be modeled with reaction kinetics. It is possible that one could come to understand their own individual faculties and calibrate a model of learning so that their experience could be optimized by understanding the related genetic and environmental constants that are present in each of the learning models presented. As presented by Šimon and Bulko, our understanding of forgetting helps us understand the operation of our own brains and how the multitude of external variables effects the ability of our brains to process information and store it (Šimon and Bulko, 2014).

This paper illustrated that conclusions related to equilibrium conditions and stability on six different learning process models are a valuable first step in the creation of a future model of the learning process. It is not the aim of this paper to presume that any of these models is fully correct, but instead present each as capturing some limited aspects of the learning process with increasing complexity. By examining each of the six models and highlighting shortcomings, it is possible to imagine the development of a future deterministic model that addresses each of these shortcomings that could be evaluated through the lens of applied mathematics. Such a theoretical model, based upon the understanding that the learning process is a biological process that can be described through chemistry could be tested with empirical data and calibrated for an individual. The existence of such a model would be incredibly useful to an individual in understanding their own limitations in knowledge acquisition but also in optimizing their learning process. The existence of such a model could also revolutionize the educational system as it would allow a means to quantify aspects of the learning process that have heretofore been impossible to measure. Such a future model is undoubtedly ambitious and rightly deserves extreme criticism that the existence is even possible. However, the authors conclude that undoubtedly there is a process in which learning occurs and although complex, it can be described in the language of mathematics and the development of such a model will begin with the analysis of equilibrium and stability. At this stage, conclusions related to learning are limited in scope and application. One of the most basic functions of humanity – our ability to learn, is not well understood. However, by examining six different learning models presented based upon a form of reaction kinetics, it is possible to understand behavior such as stability through the tools of applied mathematics and thereby illuminate certain layers of human cognition and learning. This analysis becomes the logical first step in the development of a future model of the learning process that is accurate.

References

- Aldridge, B., (1983). A mathematical model of mastery learning. *Journal of Research in Science Teaching*, 20(1), 1-17.
- Aldridge, B., Johnston, K., (1985). Examining a mathematical model of mastery learning in a classroom setting, *Journal of Research in Science Teaching*, 22(6), 543-554.
- Anderson, R., (1983). A Neuromathematical Model of Human Information Processing and its applications to Science content acquisition, *Journal of Research in Science Teaching*, 20(7), 603-620.
- Atkinson, A., Shiffrin, R., (1965). Mathematical Models for Memory and Learning. *Psychology Series: Institute for Mathematical Studies in the Social Sciences*, Technical Report No. 79, Stanford University.
- Burini, D., De Lillo, S., Gibelli, L., (2016). Collective learning modeling based on the kinetic theory of active particles, *Physics of Life Reviews*, 16, 123-139.
- Burros, R. H. (1952). Some criticisms of "A mathematical model for simple learning." *Psychological Review*, 59(3), 234–236.
- Bush, R., Mosteller, F., (2006). A Mathematical Model for Simple Learning. In: Fienberg S.E., Hoaglin D.C. (eds) *Selected Papers of Frederick Mosteller. Springer Series in Statistics*. Springer, New York, NY.
- Fey, C., (1961). An Investigation of some Mathematical Models for Learning. *Journal of Experimental Psychology*, 61(6), 455-61.
- Foss M., Liu, Y.C., (2020). Project-based learning center to bridge to students with technology, *Proceedings of 2020 Intermountain Engineering Technology and Computing (IETC)*, Orem, UT, USA.
- Foss M., Liu, Y.C., (2021). Creating solutions through project-based and experimental learning: A case study of the Concept Center, *International Journal of Engineering Education*, 37(6), 1-13.
- Fredrick, W., Walberg H., (1980). Learning as a function of time. *Journal of Educational Research*, 73, 183-194.
- Graaff, E., Kolmos, A., (2003). Characteristics of Problem-Based Learning, *International Journal of Engineering Education*, 19(5), 657-662.
- Hicklin, W.J., (1976). A model for mastery learning based on dynamic equilibrium theory. *Journal of Mathematical Psychology*, 13(1), 79–88.
- Inzlicht, M., Ben-Zeev, T., (2000). A threatening intellectual environment: Why females are susceptible to experiencing problem-solving deficits in the presence of males. *Psychological Science*, 11(5), 365–371.
- Inzlicht, M., Good, C., (2006). How environments can threaten academic performance, self-knowledge, and sense of belonging. *Stigma and Group Inequality: Social Psychological Approaches*, 129-150, Lawrence Erlbaum Associates Publishers.
- Liu, Y.C., (2017) Renovation of a mechanical engineering senior design class to an industry-tied and team-oriented course, *European Journal of Engineering Education*, 42(6), 800–811.
- Liu, Y.C., Artigue, A., Sommers J., and Chambers, T., (2011). Theo Jansen project in engineering design course and a design example, *European Journal of Engineering Education*, 36(2), 187–198.
- Liu, Y.C., Baker, F., He, W.-P., and Lai, W., (2019). Development, assessment and evaluation of laboratory experimentation for a mechanical vibrations and controls course, *International Journal of Mechanical Engineering Education*, 47(4), 315–337.

- Liu, Y.C., Baker, F., (2019). Development of vibration and control system through student projects, *Proceedings of ASEE SE Section Annual Conference*, North Carolina State University, Raleigh, NC, USA, March 10–12.
- Liu, Y.-C., Batte, J. A., Collins, Z. H., Bateman, J. N., Atkins, J., Davis, M., Salley, D., Bethel, C. L., Ball, J. E. and Archibald, C., (2017). Mechanical design, prototyping, and validation of a Martian robot mining system, *SAE International Journal of Passenger Cars – Mechanical Systems*, 10(1), 1289–1297.
- Liu, Y.C., Dou, Y.-Q., (2015). Design of an industry-tied and team-oriented course for mechanical engineering seniors, *Proceedings of ASEE SE Section Annual Conference*, University of Florida, Gainesville, FL, April 12–14.
- Liu, Y.-C., Meghat, V., and Machen, B., (2018). Design and prototyping of a debris clean and collection system for a cylinder block assembly conveying line following an engineering systems design approach, *International Journal of Design Engineering*, 8(1), 1–18.
- Liu, Y.-C., Meghat, V., and Machen, B., (2017). Design and prototyping of an in-situ robot to clean a cylinder head conveying line following an engineering systems design approach, *International Journal of Design Engineering*, 7(2), 106–122.
- Logan, J., (2013). *Applied Mathematics* (4th Ed.), Hoboken, New Jersey: John Wiley & Sons.
- Malmberg, K. Raaijmakers, J. & Shiffrin, R., (2019). 50 years of research sparked by Atkinson and Shiffrin (1968). *Memory and Cognition*, 47, 561–574.
- Ormazábal, I., Borotto, F., Astudillo, H., (2021). An agent-based model for teaching–learning processes, *Physica A: Statistical Mechanics and its Applications*, 565.
- Preece, P. (1984). Comments and Criticism Mathematical Modeling of Learning, *Journal of Research in Science Training*, 21(9), 953-955.
- Rainey, K., Dancy, M., Mickelson, R., *et al.* (2018). Race and gender differences in how sense of belonging influences decisions to major in STEM. *International Journal of STEM Education*, 5, 10.
- Schmader T., Johns, M., (2003). Converging Evidence That Stereotype Threat Reduces Working Memory Capacity. *Journal of personality and social psychology*, 85, 440-52.
- Schmidt, G., (1993). Foundations of problem-based learning: some explanatory notes. *Medical Education*, 27(5), 422-432.
- Servant-Miklos, V., (2019). Fifty Years on: A Retrospective on the World's First Problem-based Learning Programme at McMaster University Medical School, *Health Professions Education*, 5, 1, 3-12.
- Shikaa S., & Ajai, A., (2015). A Mathematical Model to Study the Human Brain Information Processing Dynamics. *American Journal of Applied Mathematics*, 3, 233.
- Šimon, J., Bulko, M., (2014). A Simple Mathematical Model of Cyclic Circadian Learning. *Journal of Applied Mathematics*, 1-9.
- Wu, X., Zhang, Y., Wu, X., and Sun, W., (2020). A Nonlinear Dynamic Model for Individual Learning with Plateau Phenomenon. In *Proceedings of the 2020 11th International Conference on E-Education, E-Business, E-Management, and E-Learning (IC4E 2020)*. Association for Computing Machinery, New York, NY, USA, 109–115.
- Youssef, N., *et al.*, (2018). The Effects of Trauma, with or without PTSD, on the Transgenerational DNA Methylation Alterations in Human Offsprings. *Brain sciences*, 8(5).